Uvažme posloupnost

$\{a_n\}_{n=1}^{\infty}$, $a_n \in \mathbb{R}$

Př: $a_n = \frac{1}{n}$ $\qquad \lim_{n\to\infty} = 0$

$a_n = n^2$ $\qquad \lim_{n\to\infty} = \infty$

$a_n = \frac{n^2+1}{2n} = \underbrace{\frac{\infty}{\infty}}_{?}$ $\qquad \lim_{n\to\infty} = ?$

Def: limita $\lim_{n\to\infty} a_n$ je rovna $a \in \mathbb{R}$ právě, když: pro $\forall$ okolí $\underbrace{O(a)}_{\text{okolí bodu } a \in \mathbb{R}}$ existuje $n_0 \in \mathbb{N}$ t.ž. $a_n \in O(a)$ $\forall n \geq n_0$

$O(a) \approx (a-\delta, a+\delta)$, $\delta > 0$ malé

příklad okolí

Př. 1

(i) $\lim\limits_{n\to\infty} \frac{2n^2+3n+1}{n+1} = \left|\frac{\infty}{\infty}\right| =$ neurčitý výraz

$= \lim\limits_{n\to\infty} \frac{2n(n+1)+(n+1)}{n+1}$

$= \lim\limits_{n\to\infty} (2n+1) = \infty$

Jinak: $\lim\limits_{n\to\infty} \frac{2n^2+3n+1}{n+1} \; \frac{/\frac{1}{n}}{/\frac{1}{n}} =$

$= \lim\limits_{n\to\infty} \frac{2n+3+\frac{1}{n}}{1+\frac{1}{n}} = \frac{\infty}{1} = \infty$

NB: $\lim\limits_{n\to\infty} \underbrace{(-1)^n}$ neexistuje

$1, -1, 1, -1, \ldots$

(ii) $\lim\limits_{n\to\infty} \underbrace{\frac{2n^2+3n+1}{3n^2+n+1}} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} =$

dělením se

zbytkem $\frac{2}{3} + \underbrace{\text{zbytek}}_{\to 0}$

$$= \lim_{n\to\infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{3 + \frac{1}{n} + \frac{1}{n^2}} = \frac{2+0}{3+0} = \frac{2}{3}$$

(iii) $$\lim_{n\to\infty} \frac{n+1}{2n^2+3n+1} \;\Big/ \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2n+3+\frac{1}{n}} = \frac{1+0}{\infty+3+0} = 0$$

Platí: 1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \pm \left(\lim\limits_{n\to\infty} b_n\right)$

• limita součtu a podílu

(iv) $$\lim_{n\to-\infty} \frac{2^n - 2^{-n}}{2^n+2^{-n}} = \lim_{n\to\infty} \frac{2^{-n}-2^{n}}{2^{-n}+2^{n}}$$

$$= \left|\frac{0-\infty}{0+\infty}\right| = \lim_{n\to\infty} \frac{2^{-2n}-1}{2^{-2n}+1} = \quad \cdot \frac{2^{-n}}{2^{-n}}$$

$$= \frac{0-1}{0+1} = -1$$

(v) $$\lim_{n\to\infty} \frac{\sqrt{4n^2+n}}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{4+\frac{1}{n}}}{1+\frac{1}{n}}$$

$$= \frac{\sqrt{4+0}}{1+0} = 2$$

(vi) $\lim\limits_{n\to\infty}\left(\sqrt{4n^2+n}-2n\right) = |\infty-\infty|$ neurčitý výraz

$$= \lim_{n\to\infty}\left(\sqrt{4n^2+n}-2n\right)\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n}$$

$$= \lim_{n\to\infty}\frac{(\cancel{4n^2}+n)-\cancel{4n^2}}{\sqrt{4n^2+n}+2n} = \lim_{n\to\infty}\frac{n}{\sqrt{4n^2+n}+2n} \quad \cdot\frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{4+\frac{1}{n}}+2} = \frac{1}{2+2} = \underline{\underline{\frac{1}{4}}}$$

Př. 2:

(i) $c=1 \to$ platí

$c>1$: $a_n = \sqrt[n]{c}$ je klesající posloupnost pro $n\to\infty$,

$a_n > 1$

Sporem předp., že $\lim\limits_{n\to\infty}\sqrt[n]{c} = 1+\varepsilon$,

$\varepsilon > 0$

Zvolme okolí $O(1+\varepsilon) = \left(1+\varepsilon-\frac{\varepsilon^2}{4}, 1+\varepsilon+\frac{\varepsilon^2}{4}\right)$

pak $\exists n_0 \in \mathbb{N}: \forall n \geq n_0:$ $\overset{\Downarrow}{a_n}$

$$1+\varepsilon-\frac{\varepsilon^2}{4} \leq \sqrt[n]{c} \leq \underbrace{1+\varepsilon+\frac{\varepsilon^2}{4}}_{\left(1+\frac{\varepsilon}{2}\right)^2}$$

$$\sqrt[n]{c} \leq \left(1+\frac{\varepsilon}{2}\right)^2 \;/\; (\;)^{1/2}$$

$$\underbrace{\sqrt[2n]{c}}_{a_{2n}} \leq 1+\frac{\varepsilon}{2}$$

$$a_{2n} \leq 1+\frac{\varepsilon}{2} < 1+\varepsilon$$

$$a_{2n} < 1+\varepsilon \quad \text{spor}$$

$0 < c < 1 \rightsquigarrow \frac{1}{c} > 0$ . . . .

(ii) $\underbrace{\lim_{n\to\infty} \sqrt[n]{n} = 1}$

$a_n > 0$

Položme $\underbrace{\sqrt[n]{n}}_{>1 \text{ pro } n \geq 2} = 1+a_n \;/ (\;)^n$

$$n = (1+a_n)^n$$

$$n = 1 + \binom{n}{1} a_n + \binom{n}{2} a_n^2 + \dots \binom{n}{n-1} a_n^{n-1} + a_n^n$$

$$n \geq \binom{n}{2} a_n^2 = \frac{1}{2} n(n-1) a_n^2 \qquad \left/ \frac{1}{2n(n-1)}\right.$$

$$\frac{2}{n-1} \geq a_n^2$$

$$\sqrt{\frac{2}{n-1}} \geq a_n$$

$$\Rightarrow \quad 0 \leq a_n \leq \sqrt{\frac{2}{n-1}}$$

PMD $n \to \infty$ $\quad 0 \leq \lim_{n\to\infty} a_n \leq \underbrace{\lim_{n\to\infty} \sqrt{\frac{2}{n-1}}}_{0}$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

„Věta o třech limitách"

jestliže $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = D$

a platí $b_n \leq a_n \leq c_n$

$$\Rightarrow \lim_{n\to\infty} a_n = D$$

Pozn.: $1 \le \sqrt[n]{c} \le \sqrt[n]{n}$, $c > 1$

$\downarrow$

pro velká n

$$1 \le \underbrace{\lim_{n\to\infty} \sqrt[n]{c}}_{=1} \le \underbrace{\lim_{n\to\infty} \sqrt[n]{n}}_{=1}$$

Př 2.3:

$$\lim_{n\to\infty} \frac{\sqrt{n^3 - 11n^2 + 2} + \sqrt[5]{n^7 - 2n^5 - n^3 - n + \sin^2 n}}{2 - \sqrt[3]{5n^4 + 2n^3 + 5}} \cdot \frac{n^{-\frac{4}{3}}}{n^{-\frac{4}{3}}}$$

$$\lim_{n\to\infty} \frac{\sin^2 n}{2 - \sqrt[3]{5n^4 + 2n^3 + 5}} = \frac{\langle\ \rangle^{[0,1]}}{-\infty} =$$

čit: $n^{\frac{3}{2}}, n^{\frac{7}{5}}, n$. $\Big\}$ $\frac{3}{2}$ největší exponent

jm: $n^{\frac{4}{3}}$

$$= \lim_{n\to\infty} \frac{\sqrt[6]{(n^3 - 11n^2 + 2)^3 \cdot n^{-8}} + \sqrt[15]{(n^7 - 2n^5 - n^3)^3 n^{-20}} \ldots}{2n^{-4/3} - \sqrt[3]{5 + 2n^{-1} + 5n^{-4}}}$$

$-n^{-\frac{1}{3}}$

$$= \frac{\infty + \infty - 0}{0 - \sqrt[3]{5+0+0}} = -\infty$$

$$n^{-8} = \left(n^{-\frac{8}{3}}\right)^3$$

$$= \lim_{n\to\infty} \frac{\sqrt[6]{\left(n^{3-\frac{8}{3}} - 11 n^{2-\frac{8}{3}} + 2n^{-\frac{8}{3}}\right)} + \sqrt{- \;\dots}}{- \; - \;\dots}$$

Pr. 4

(i) $$\lim_{n\to\infty} \frac{3^n + (-2)^{n+1}}{3^{n-2} - 2^{2n-1}} =$$

$$= \lim_{n\to\infty} \frac{3^n - 2\cdot(-2)^n}{\left(\frac{1}{9}\right)3^n - \frac{1}{2}\left(4^n\right)} = \left|\frac{\infty \pm \infty}{\infty - \infty}\right|$$

$$= \lim_{n\to\infty} \frac{3^n\left[1 - 2\left(-\frac{2}{3}\right)^n\right]}{4^n\left[\frac{1}{9}\left(\frac{3}{4}\right)^n - \frac{1}{2}\right]} =$$

$$= \underbrace{\lim_{n\to\infty}\left(\frac{3}{4}\right)^n}_{\to 0} \cdot \underbrace{\lim_{n\to\infty} \frac{1 - 2\left(-\frac{2}{3}\right)^n}{\frac{1}{9}\left(\frac{3}{4}\right)^n - \frac{1}{2}}}_{\to \frac{1-0}{0-\frac{1}{2}} = -2}$$

$$= 0 \cdot (-2) = 0$$

(ii) $\lim\limits_{n\to\infty} \underbrace{\overbrace{(1+\sin n)}^{a_n}}_{\in [0,2]}{}^{n}$ neexistuje

Z posloupnosti a_n mohu vybrat podposloupnost

- $b_n > 1{,}01 \Rightarrow \lim\limits_{n\to\infty} b_n = \infty$

- $c_n < 0{,}99 \Rightarrow \lim\limits_{n\to\infty} c_n = 0$