Pr 3.8 $f(x) = \frac{\ln(cx)}{\sqrt{x}}$

v bodě [1, 0] tečna procházející bodem [?, 2]

- směrnice na odvodnice

$f'(1) = 2$

- $f(1) = 0 \rightsquigarrow \frac{\ln(c \cdot 1)}{1} = 0$

$\Rightarrow c = 1$

$$f'(x) = \frac{\frac{1}{cx} \cdot c \cdot \sqrt{x} - \ln(cx) \frac{1}{2} x^{1-\frac{1}{2}}}{x}$$

$$= \frac{\frac{1}{\sqrt{x}} - \frac{1}{2} \ln(cx) \frac{1}{\sqrt{x}}}{x} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} =$$

$$= \frac{2 - \ln(cx)}{2x\sqrt{x}}$$

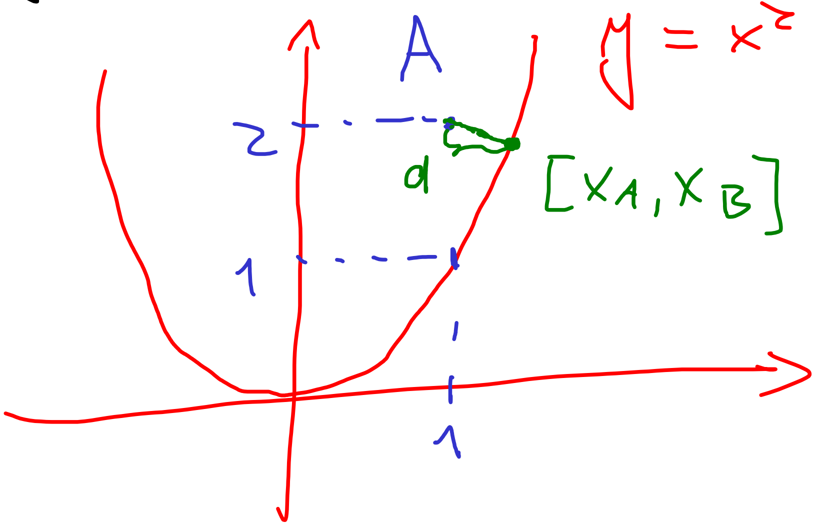
$$f'(1) = \frac{2 - \ln c}{2} = 2 \Rightarrow 2 - \ln c = 4$$

$\ln c = -2$

$c = \frac{1}{e^2}$

Nejde:

3.10: Hledáme bod $[x_A, x_B]$ paraboly $y = x^2$, který je nejblíž bodu $A = [1,2]$.



nechť (x,y) je bod na parabole $\rightsquigarrow$ jeho vzdálenost od bodu A je

Bod na parabole je $[x, x^2]$

$$\sqrt{(x-1)^2 + (x^2-2)^2}$$

↳ hľadáme minimum tejto

$$g(x) = (x-1)^2 + (x^2-2)^2$$

Nájdeme lokálne extrémy

$$g'(x) = 2(x-1) + 2(x^2-2)\cdot 2x = 0$$

$$2x(x^2-2) + x - 1 = 0$$

$$2x^3 - 3x - 1 = 0$$

$x = -1$:

	2	0	-3	-1
-1	2	-2	-1	0

$$2x^3 - 3x - 1 = (x+1)\underbrace{(2x^2-2x-1)}$$

Celkom 3 korene

$$x_{1,2} = \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \begin{cases} \frac{1}{2} + \frac{\sqrt{3}}{2} \\ \frac{1}{2} - \frac{\sqrt{3}}{2} \end{cases}$$

(stacionární body)

$$[-1,1],\ \Big[\underbrace{\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}}_{>1},\ \Big(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\Big)^2\Big],\ \Big[\underbrace{\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}}_{<1}\Big(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\Big)^2\Big]$$

$$x_A = \frac{1}{2}+\frac{\sqrt{3}}{2}$$

$$g''(x) = 6x^2 - 3$$

$g''(-1) > 0 \rightsquigarrow$ lok. min

$g''\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right) > 0 \rightsquigarrow$ lok. min

$$g''\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right) = 6\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)^2 - 3$$

$$= 6\left(\frac{1}{4} - \frac{\sqrt{3}}{2} + \frac{3}{4}\right) - 3$$

$$= 6\left(1 - \frac{\sqrt{3}}{2}\right) - 3 = \underbrace{6 - 3\sqrt{3} - 3}_{<0}$$

lok. max

$$g(-1) = (-1-1)^2 + (1-2)^2 = 4+1 = 5$$

$$g\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2}+\frac{\sqrt{3}}{2}-1\right)^2 + \left[\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)^2-2\right]^2$$

$$= \ldots < 5$$

4.1 (i) $\lim_{x \to 0} \frac{\sin(2x) - 2\sin x}{2e^x - x^2 - 2x - 2} =$

type $\frac{0}{0}$

$$= \lim_{x \to 0} \frac{2\cos(2x) - 2\cos x}{2e^x - 2x - 2} = \left| \frac{2-2}{2-2} = \frac{0}{0} \right|$$

l'Hospitalovo pravidlo

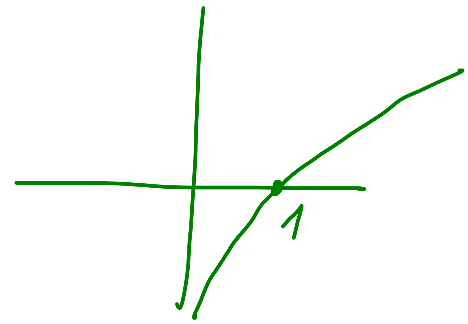
$$= \lim_{x \to 0} \frac{-2\cdot 2\cdot \sin(2x) + 2\sin x}{2e^x - 2} = \left|\frac{0}{0}\right|$$

$$= \lim_{x \to 0} \frac{-8\cos(2x) + 2\cos x}{2e^x} = \frac{-8+2}{2}$$

$$= -\frac{6}{2} = -3$$

(iii) $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$? ?

• $\lim_{x \to 1^-} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) =$



$$= |-\infty - (-\infty)|$$

$$= |-\infty + \infty| \leftarrow$$ neurčitý typ

• $\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$

typu $\infty - \infty$

$$\lim_{x \to 1} \frac{x \ln x - (x-1)}{(x-1)\ln x} \quad \text{typu } \frac{\pm 0}{0}$$

$$\lim_{x \to 1^+} \frac{x \ln x - (x-1)}{(x-1)\ln x} \overset{l'H}{=}$$

$$= \lim_{x \to 1^+} \frac{\left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) - 1}{1 \cdot \ln x + \frac{x-1}{x}} \cdot \frac{x}{x} =$$

$$= \lim_{x \to 1^+} \frac{x \ln x}{x \ln x + x - 1} = \left| \frac{0}{0} \right|$$

L'H

$$= \lim_{x \to 1^+} \frac{\ln x + x \cdot \frac{1}{x}}{\ln x + x \cdot \frac{1}{x} + 1} =$$

$$= \lim_{x \to 1^+} \frac{\ln x + 1}{\ln x + 2} = \underline{\underline{\frac{1}{2}}}$$

<u>Pozn.</u>: $\lim_{x \to 1^-} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \dots = \frac{1}{2}$

Tedy: $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \frac{1}{2}$

(iv) $\lim_{x \to 1^+} \ln x \cdot \ln (x-1) =$

$$= \left| 0 \cdot (-\infty) \right| =$$

$$= \lim_{x \to 1^+} \frac{\ln x}{\frac{1}{\ln (x-1)}} = \left| \frac{0}{0} \right| =$$

L'H

$$= \lim_{x \to 1+} \frac{\frac{1}{x}}{-(\ln (x-1))^{-2} \cdot \frac{1}{x-1}} \cdot \frac{x(x-1)}{x(x-1)}$$

$$= \lim_{x \to 1^+} \frac{x-1}{-\frac{x}{\ln(x-1)^2}} =$$

$$= \lim_{x \to 1^+} - \frac{(x-1)\left(\ln(x-1)\right)^2}{x} = \left| \frac{0\,(+\infty)}{1} \right|$$

$$= \lim_{x \to 1^+} - \frac{\left(\ln(x-1)\right)^2}{\frac{x}{x-1}} = \left| - \frac{0}{0} \right|$$

$$= \lim_{x \to 1^+} - \frac{2\ln(x-1) \cdot \frac{1}{x-1}}{\frac{(x-1)-x}{(x-1)^2}} \frac{(x-1)^2}{(x-1)^2}$$

$$= \lim_{x \to 1^+} - \frac{2(x-1)\ln(x-1)}{-1} =$$

$$= \lim_{x \to 1^+} 2(x-1)\ln(x-1) = |0 \cdot (-\infty)|$$

$$= \lim_{x \to 1^+} 2 \frac{\ln(x-1)}{\frac{1}{x-1}} = \left| \frac{0}{0} \right|$$

$$= 2 \lim_{x \to 1^+} \frac{\frac{1}{x-1}}{-(x-1)^{-2}} =$$

$$= > \lim_{x \to 1^+} -(x-1) = 0$$

Výsledek je: $\lim_{x \to 1^+} \ln(x-1)\ln(x) = 0$

(v) $\lim_{x \to 0^+} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ typu 1^∞

$$= \lim_{x \to 0^+} e^{\ln\left(\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}\right)} =$$

$$= \lim_{x \to 0^+} e^{\frac{1}{x^2} \ln \frac{\sin x}{x}} =$$

$$= e^{\lim_{x \to 0^+} \frac{1}{x^2}\left(\ln \frac{\sin x}{x}\right)} \quad \infty \cdot 0$$

$= \dots$

Př. 4.2: $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) =$

$$= \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right)$$

$+ \ldots\ldots$

$$S_m = \sum_{n=1}^{m} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{m+1}}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \lim_{m \to \infty} S_m = 1$$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \ln(n+1) - \ln n$$

$$= (\cancel{\ln 2} - \ln 1) + (\cancel{\ln 3} - \ln 2) + (\ln 4 - \ln 3) + \ldots$$

$$S_m = \sum_{n=1}^{m} \ln\left(\frac{n+1}{n}\right) = -\ln 1 + \ln(m+1) = \ln(m+1)$$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \infty$$

$$\sum_{n=1}^{\infty} \frac{n^2+1}{3n} = \sum_{n=1}^{\infty} \Big(\frac{n}{3} + \underbrace{\frac{1}{3n}}_{\infty} \Big)$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{n}{3}}_{=\infty} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{3n}}_{=\infty}$$