51. (i) $\sum_{n=1}^{\infty}\left(\frac{3}{4^{2n-1}}+\frac{2}{4^{2n}}\right)=$

$=\sum_{n=1}^{\infty}\left(4\cdot\frac{3}{4^{2n}}+\frac{2}{4^{2n}}\right)$

$=14\cdot\left(\sum_{n=1}^{\infty}\frac{1}{4^{2n}}\right)=14\left(\sum_{n=1}^{\infty}\frac{1}{16^{n}}\right)$

$=14\cdot\frac{\frac{1}{16}}{1-\frac{1}{16}}=14\cdot\frac{\frac{1}{16}}{\frac{15}{16}}=\underline{\underline{\frac{14}{15}}}$

(ii) $\sum_{n=1}^{\infty}\frac{n}{3^{n}}=\lim_{m\to\infty}S_m$

$S_m=\sum_{n=1}^{m}\frac{n}{3^{n}}$

$$S_n = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \dots + \frac{n}{3^n}$$

$$\frac{1}{3}S_n = \frac{1}{3^2} + \frac{2}{3^3} + \dots + \frac{n-1}{3^n} + \frac{n}{3^{n+1}}$$

$$\underbrace{S_n - \frac{1}{3}S_n}_{\frac{2}{3}S_n} = \underbrace{\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n}}_{\text{geom. řada}} - \frac{n}{3^{n+1}}$$

$$\lim_{n\to\infty}\left(\frac{2}{3}S_n\right) = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\right)$$

$$\parallel$$

$$\frac{2}{3}\left(\lim_{n\to\infty} S_n\right) = \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}} = \frac{1}{3}\cdot\frac{1}{\frac{2}{3}} = \frac{1}{2}$$

$$\lim_{n\to\infty} S_n = \underline{\underline{\frac{3}{4}}}$$

(iii) $$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+4)} =$$

$A, B \in \mathbb{R}$

$$\frac{1}{(3n+1)(3n+4)} = \frac{A}{3n+1} + \frac{B}{3n+4}$$

$$= \frac{A(3n+4) + B(3n+1)}{(3n+1)(3n+4)}$$

$$\Rightarrow \quad 1 = A(3n+4) + B(3n+1)$$

$$1 = 3(A+B)\,n + (4A+B)$$

$$\left.\begin{aligned} A+B &= 0 \\ 4A+B &= 1 \end{aligned}\right\} \quad \begin{aligned} B &= -A \\ 4A - A &= 1 \end{aligned}$$

$$\boxed{\begin{aligned} A &= \frac{1}{3} \\ B &= -\frac{1}{3} \end{aligned}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3} \frac{1}{3n+1} - \frac{1}{3} \cdot \frac{1}{3n+4} \right) =$$

$$= \frac{1}{3} \left(\underbrace{1 + \frac{1}{4} + \frac{1}{7} + \dots}_{m} \;-\; \underbrace{\frac{1}{4} - \frac{1}{7} - \frac{1}{10} \dots}_{m} \right)$$

$$\underbrace{\frac{1}{3}\left(1 - \frac{1}{3n+4}\right)}_{\to \frac{1}{3} \text{ pro } n \to \infty} = S_n$$

S_n

n-tý částečný součet

$S_{\text{řady}} = \frac{1}{3}$

Pozor: vozdělení na dvě

sumy dává

$$\frac{1}{3}\underbrace{\sum \frac{1}{3n+1}}_{=\infty} - \frac{1}{3}\underbrace{\sum \frac{1}{3n+4}}_{=\infty}$$

Víte $\sum \frac{1}{n} = \infty$

harmonická řada

$$\sum_{n=0}^{\infty} \frac{1}{3n+1} =$$

$\Vert$

posunutá harm. řada

$$\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{3}}$$

$$\frac{1}{n+\frac{1}{3}} = \frac{1}{(n+1)-\frac{2}{3}} > \frac{1}{n+1} \qquad \forall n$$

$$\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{3}} > \sum_{n=0}^{\infty}\frac{1}{n+1} = \infty$$

srovnávací kritérium

kon / div

PMO

5.2 (ii) $\displaystyle\sum_{n=1}^{\infty}\overbrace{\frac{1}{(n+1)3^n}}^{a_n} =$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| =$$

podílové kritérium

$$\underset{\lim}{=}\frac{\frac{1}{(n+2)3^{n+1}}}{\frac{1}{(n+1)3^n}} = \lim_{n\to\infty}\frac{(n+1)3^n}{(n+2)3^{n+1}} =$$

$$= \lim_{n\to\infty}\frac{n+1}{3(n+2)} = \frac{1}{3}\underbrace{\lim_{n\to\infty}\frac{n+1}{n+2}}_{=1} =$$

$= \frac{1}{3} < 1$ $\qquad = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}}$

Závěr: řada konverguje

Jinak: odmocninové kritérium

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(n+1)3^n}} =$$

$$= \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt[n]{n+1}}_{=1} \cdot 3} = \frac{1}{3}$$

Ještě jinak: srovnávací krit.

$$\sum \frac{1}{(n+1)3^n} < \underbrace{\sum \frac{1}{3^n}}_{\text{konverguje}}$$

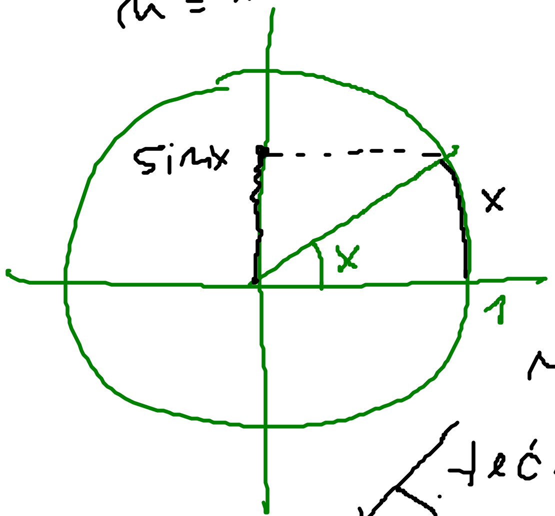
(iii) $\displaystyle \sum_{n=1}^{\infty} \frac{n^2+1}{n^3} = \sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n^3}\right) =$

div. $\displaystyle = \left(\sum_{n=1}^{\infty} \frac{1}{n}\right) + \left(\sum_{n=1}^{\infty} \frac{1}{n^3}\right)$

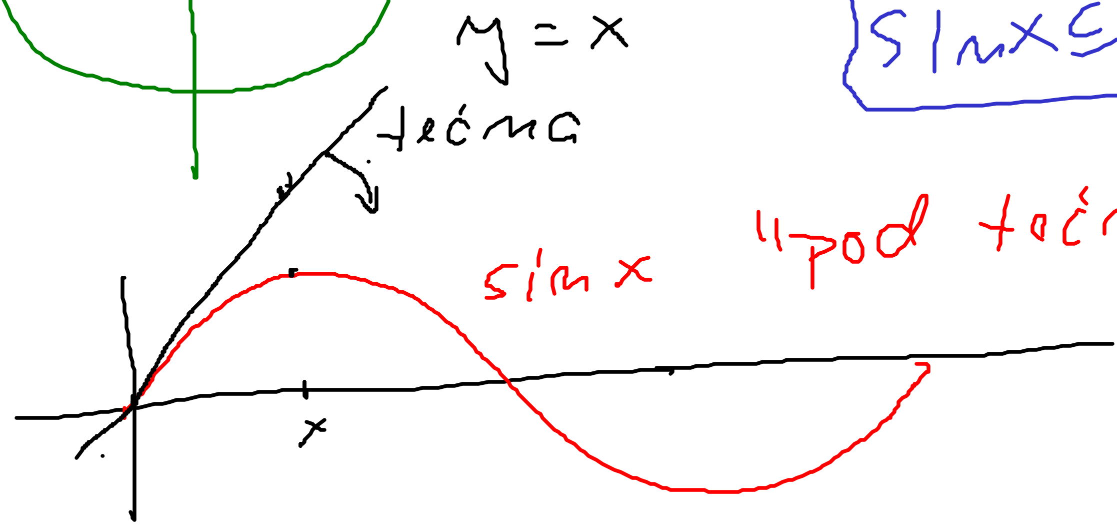
(iv) $\sum_{n=1}^{\infty} \sin\frac{\pi}{n^2} =$

$\sin x < x$

$x \in \left(0, \frac{\pi}{2}\right)$



$$\boxed{\begin{array}{l} \dfrac{\sin x}{\cos x} \le 1 \\ \sin x \le \cos x \le 1 \end{array}}$$



Směrnice tečny $(\sin x)'$ v bodě $x = 0$

$\cos 0 = 1$

$\underbrace{\sin\frac{\pi}{n^2} < \frac{\pi}{n^2}}_{}$ pro $n \geq 3$

konverguje

$\sum \frac{1}{n}$ div.

$\sum \frac{1}{n^2}$ konv. (dozvíte se později)

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)!\, n^n}{n!\,(n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\, n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n =$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)-1}{n+1}\right)^n = \lim_{n\to\infty} \left(1-\frac{1}{n+1}\right)^n$$

$= \ldots$

$$\lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} =$$

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = 2.7$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

konvergego

(vi) $\sum_{n=1}^{\infty} (-1)^{n+1} \ln\left(1+\frac{1}{n}\right)$

alternující $\Rightarrow$ konv. ??

Leibnizovo krit.

$\sum a_n$ konv. $\Leftrightarrow \lim_{n\to\infty} a_n = 0$

střídá znaménko

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \underbrace{\ln\left(1+\frac{1}{n}\right)}_{\to 0 \text{ pro } n\to\infty} = 0$

konv.

Př 5.3 Mocninné řady

(i) $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$

poloměr konvergence r

$\Rightarrow x\in(-r,r)$ řada konv.

$x \in (-\infty, -r) \cup (r, \infty)$ řad. div.

$\left.\begin{array}{l} x = -r \\ x = r \end{array}\right\}$ jak kdy :-(

$$r = \frac{1}{\lim\limits_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|}$$

$a_n = \frac{2^n}{n}$

$\sum a_n x^n$

obecněji

$$r = \frac{1}{\limsup\limits_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|}$$

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} =$$

$$= \lim_{n \to \infty} 2 \cdot \frac{n}{n+1} = 2$$

$$\Rightarrow \boxed{r = \frac{1}{2} \Rightarrow \text{konv. } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \quad \text{div. } x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)}$$

Vyšetříme krajní body:

$x = -r \rightsquigarrow \displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n =$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{konv.}$$

$x = r \rightsquigarrow \displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}\cdot\left(\frac{1}{2}\right)^n = \sum \frac{1}{n}$

div.

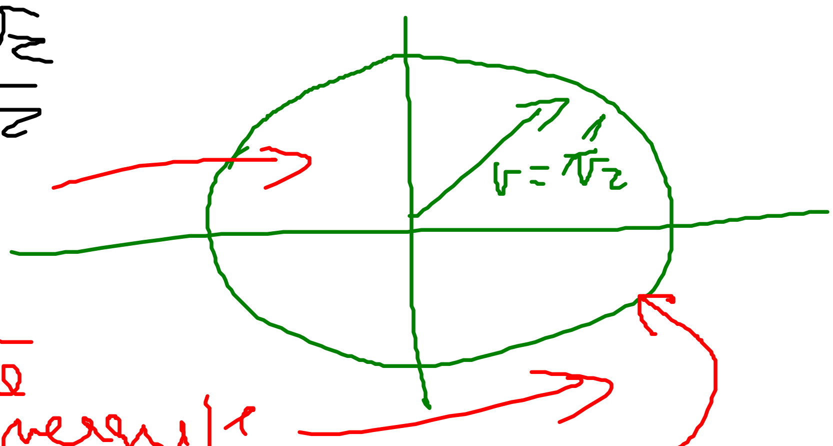
$$\text{(ii)} \sum_{n=1}^{\infty} \frac{1}{(1+i)^n}\, x^n \quad , \quad x \in \mathbb{C}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(1+i)^n|} =$$

$$= \lim_{n\to\infty} \sqrt[n]{|1+i|^n} = |1+i| = \sqrt{2}$$

$$r = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

uvnitř konv.



vně diverguje

na hranici složité

(iii) $\sum_{n=1}^{\infty} (-4n)^n x^n$, $a_n = (-4n)^n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(-4n)^n|}$

$= \lim_{n\to\infty} \sqrt[n]{(4n)^n} = \lim_{n\to\infty} 4n$
$= \infty$

$\Rightarrow r = 0$

$\Rightarrow$ konverguje jen pro $x = 0$

$\sum a_n x^n$