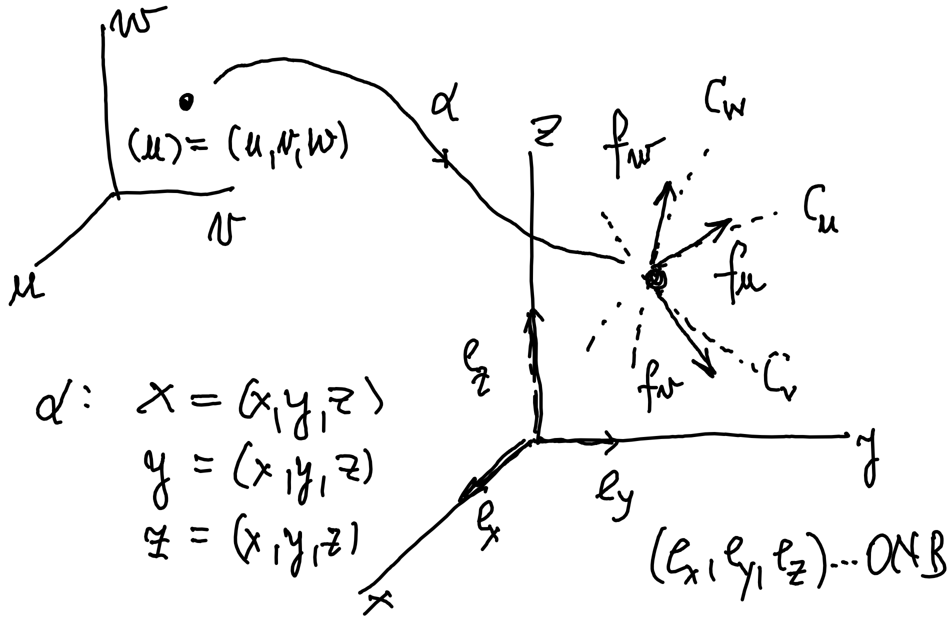
# Diferenciální operátory v křivočarých souřadnicích




$\alpha:\quad x = (x, y, z)$

$\quad\;\; y = (x, y, z)$

$\quad\;\; z = (x, y, z)$

$(e_x, e_y, e_z) \ldots$ ONB

Souřadnicové křivky a tečné vektory

$$f_u \sim \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right),\ f_v \sim \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right),\ f_w \sim \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right)$$

normování $f_u = |f_u| e_u$

$$f_u = \frac{\partial x}{\partial u} e_x + \frac{\partial y}{\partial u} e_y + \frac{\partial z}{\partial u} e_z,\ f_v = \ldots,\ f_w = \ldots$$

Nalezení duální báze k ONB $(e_u, e_v, e_w)$:

Platí: $f_u = \frac{\partial}{\partial u}$ (označení); $f^u = du$, $f^v = dv$, $f^w = dw$

$$f_u = |f_u| e_u, \quad f_v = |f_v| e_v, \quad f_w = |f_w| e_w$$

$$1 = f^u(f_u) = f^u(|f_u| e_u) = |f_u| f^u(e_u) =$$

$$= |f_u| du(e_u) \Rightarrow \begin{array}{ll} e^u = |f_u| du & e^x = dx \\ e^v = |f_v| dv & e^y = dy \\ e^w = |f_w| dw & e^z = dz \end{array}$$

$$\underline{\omega_0 = dx \wedge dy \wedge dz = e^u \wedge e^v \wedge e^w = |f_u| \cdot |f_v| \cdot |f_w| \, du \wedge dv \wedge dw}$$

Vektorové pole $F \sim (F_x, F_y, F_z) \sim (F_u, F_v, F_w)$

$$\omega_F^{(1)} = F_x dx + F_y dy + F_z dz = F_u e^u + F_v e^v + F_w e^w$$

$$\omega_F^{(2)} = F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy$$
$$= F_u e^v \wedge e^w + F_v e^w \wedge e^u + F_w e^u \wedge e^v$$

$$\omega_F^{(3)} = (F_x + F_y + F_z) dx \wedge dy \wedge dz =$$
$$= (F_u + F_v + F_w) e^u \wedge e^v \wedge e^w$$

# Divergence

$$i_F \omega_0 = i_F (dx \wedge dy \wedge dz) =$$

$$= F_x \, dy \wedge dz - F_y \, dx \wedge dz + F_z \, dx \wedge dy = \omega_F^{(2)}$$

$$d\omega_F^{(2)} = \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx \wedge dy \wedge dz$$

$$\Rightarrow \quad d\omega_F^{(2)} = i_F \omega_0 = \operatorname{div} F \, dx \wedge dy \wedge dz$$

$$= \operatorname{div} F \Big|_{(u,v,w)} e^u \wedge e^v \wedge e^w$$

$$i_F \omega_0 = i_F \, e^u \wedge e^v \wedge e^w =$$

$$= F_u \, e^v \wedge e^w + F_v \, e^w \wedge e^u + F_w \, e^u \wedge e^v =$$

$$F_u |f_v||f_w| \, dv \wedge dw + F_v |f_u||f_w| \, dw \wedge du +$$

$$+ F_w |f_u||f_v| \, du \wedge dv$$

$$d i_F \omega_0 = \left( \frac{\partial (F_u |f_v||f_w|)}{\partial u} + \frac{\partial (F_v |f_u||f_w|)}{\partial v} + \frac{\partial (F_w |f_u||f_v|)}{\partial w} \right) \cdot$$

$$\cdot \, du \wedge dv \wedge dw =$$

$$d i_F w_0 = \left( \frac{\partial(F_u |f_v||f_w|)}{\partial u} + \frac{\partial(F_v |f_u||f_w|)}{\partial v} + \frac{\partial(F_w |f_u||f_v|)}{\partial w} \right) \cdot$$

$$\underbrace{\frac{1}{|f_u|} e^u \wedge \frac{1}{|f_v|} e^v \wedge \frac{1}{|f_w|} e^w}_{du \wedge dv \wedge dw} =$$

$$= \frac{1}{|f_u||f_v||f_w|} \cdot \left[ \frac{\partial(F_u |f_v||f_w|)}{\partial u} + \right.$$

$$\left. + \frac{\partial(F_v |f_u||f_w|)}{\partial v} + \frac{\partial(F_w |f_u||f_v|)}{\partial w} \right] e^u \wedge e^v \wedge e^w$$

$$\Rightarrow \operatorname{div} F = \left[ \frac{\partial(F_u |f_v||f_w|)}{\partial u} + \frac{\partial(F_v |f_u||f_w|)}{\partial v} + \right.$$

$$\left. + \frac{\partial(F_w |f_u||f_v|)}{\partial w} \right] \cdot \frac{1}{|f_u||f_v||f_w|}$$

$$\operatorname{div} F\Big|_{(u,v,w)} = \left( \frac{\partial(F_u |f_v||f_w|)}{\partial u} + \frac{\partial(F_v |f_u||f_w|)}{\partial v} + \frac{\partial(F_w |f_u||f_v|)}{\partial w} \right)$$

Kontrolní příklad sférické souřadnice

$x = r\sin\vartheta\cos\varphi,\ y = r\sin\vartheta\sin\varphi,\ z = r\cos\vartheta$

$f_r \sim (\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta)$

$f_\vartheta \sim (r\cos\vartheta\cos\varphi,\ r\cos\vartheta\sin\varphi,\ -r\sin\vartheta)$

$f_\varphi \sim (-r\sin\vartheta\sin\varphi,\ r\sin\vartheta\cos\varphi,\ 0)$

$|f_r| = 1,\ |f_\vartheta| = r,\ |f_\varphi| = r\sin\vartheta$

$e^r = dr,\ e^\vartheta = r\,d\vartheta,\ e^\varphi = r\sin\vartheta\,d\varphi$

$$\operatorname{div} F = \left[\frac{\partial(F_r \cdot r^2\sin\vartheta)}{\partial r} + \frac{\partial(F_\vartheta r\sin\vartheta)}{\partial\vartheta} + \frac{\partial(F_\varphi \cdot r)}{\partial\varphi}\right]\frac{1}{r^2\sin\vartheta}$$

$$\frac{1}{r^2\sin\vartheta}\left(\frac{\partial F_r}{\partial r} r^2\sin\vartheta + \frac{\partial F_\vartheta}{\partial\vartheta} r\sin\vartheta + \frac{\partial F_\varphi}{\partial\varphi} r + F_r(2r\sin\vartheta) + F_\vartheta r\cos\vartheta\right)$$

# Parabolické súradnice

$$x = uv\cos\varphi,\quad y = uv\sin\varphi,\quad z = \tfrac{1}{2}(u^2 - v^2)$$

$$f_u \sim (v\cos\varphi,\ v\sin\varphi,\ u),\quad |f_u| = \sqrt{u^2+v^2}$$

$$f_v \sim (u\cos\varphi,\ u\sin\varphi,\ -v),\quad |f_v| = \sqrt{u^2+v^2}$$

$$f_\varphi \sim (-uv\sin\varphi,\ uv\cos\varphi,\ 0),\quad |f_\varphi| = uv$$

$$\operatorname{div} F\Big|_{(u,v,\varphi)} = \left[ \frac{\partial\left(F_u \cdot uv\sqrt{u^2+v^2}\right)}{\partial u} + \frac{\partial\left(F_v \cdot uv\sqrt{u^2+v^2}\right)}{\partial v} + \frac{\partial\left(F_v\,(u^2+v^2)\right)}{\partial\varphi} \right] \frac{1}{uv(u^2+v^2)}$$

$$= \frac{\partial F_u}{\partial u}\cdot uv\sqrt{u^2+v^2} + \frac{\partial F_v}{\partial v}\,uv\sqrt{u^2+v^2} + \frac{\partial F_v}{\partial\varphi}(u^2+v^2)$$

$$+ F_u\left[ v\sqrt{u^2+v^2} + \frac{u^2 v}{\sqrt{u^2+v^2}} \right] + F_v\left[ u\sqrt{u^2+v^2} + \frac{uv^2}{\sqrt{u^2+v^2}} \right]$$

$$\boxed{\operatorname{div} F = \left\{ uv\sqrt{u^2+v^2}\left(\frac{\partial F_u}{\partial u} + \frac{\partial F_v}{\partial v}\right) + (u^2+v^2)\frac{\partial F_v}{\partial\varphi} + \frac{1}{\sqrt{u^2+v^2}}\left[ F_u(2u^2v + v^3) + F_v(2uv^2 + u^3) \right] \right\} \frac{1}{uv(u^2+v^2)}}$$

## Rotace

$$d\omega^{(1)}_F = \omega^{(2)}_{\mathrm{rot}F}$$

$$\omega^{(1)}_F = F_x\,dx + F_y\,dy + F_z\,dz = F_u\,du + F_v\,dv + F_w\,dw$$

$$= F_u|f_u|\,du + F_v|f_v|\,dv + F_w|f_w|\,dw$$

$$d\omega^{(1)}_F = \left(\frac{\partial F_w|f_w|}{\partial v} - \frac{\partial F_v|f_v|}{\partial w}\right) dv\wedge dw +$$

$$+ \left(\frac{\partial F_u|f_u|}{\partial w} - \frac{\partial F_w|f_w|}{\partial u}\right) dw\wedge du +$$

$$+ \left(\frac{\partial F_v|f_v|}{\partial u} - \frac{\partial F_u|f_u|}{\partial v}\right) du\wedge dv$$

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$$d\omega^{(1)}_F = \left(\frac{\partial F_w|f_w|}{\partial v} - \frac{\partial F_v|f_v|}{\partial w}\right)\frac{1}{|f_v||f_w|}\, e^v\wedge e^w +$$

$$+ \left(\frac{\partial F_u|f_u|}{\partial w} - \frac{\partial F_w|f_w|}{\partial u}\right)\frac{1}{|f_u||f_w|}\, e^w\wedge e^u +$$

$$+ \left(\frac{\partial F_v|f_v|}{\partial u} - \frac{\partial F_u|f_u|}{\partial v}\right)\frac{1}{|f_u||f_v|}\, e^u\wedge e^v$$

$$\omega^{(2)}_{\mathrm{rot}F} = \left( r\sin\vartheta \frac{\partial F_\varphi}{\partial \vartheta} + F_\varphi r\cos\vartheta - r\frac{\partial F_\vartheta}{\partial \varphi} \right) \frac{e^\vartheta \wedge e^\varphi}{r^2 \sin\varphi}$$

SFER.

$$+ \left( \frac{\partial F_r}{\partial \varphi} - r\sin\vartheta \frac{\partial F_\varphi}{\partial r} - F_\varphi \sin\vartheta \right) \frac{e^\varphi \wedge e^r}{r \sin\varphi}$$

$$+ \left( r\frac{\partial F_\vartheta}{\partial r} + F_\vartheta - \frac{\partial F_r}{\partial \vartheta} \right) \frac{e^r \wedge e^\vartheta}{r}$$

Složky rotace jsou v závorkách

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Parabolické souřadnice ... dopočítat

$$\omega^{(2)}_{\mathrm{rot}F} = \left( \frac{\partial F_\varphi |f_\varphi|}{\partial v} - \frac{\partial F_v |f_v|}{\partial \varphi} \right) dv \wedge d\varphi$$

$$+ \left( \frac{\partial F_u |f_u|}{\partial \varphi} - \frac{\partial F_\varphi |f_\varphi|}{\partial u} \right) d\varphi \wedge du \; +$$

$$+ \left( \frac{\partial F_v |f_v|}{\partial u} - \frac{\partial F_u |f_u|}{\partial v} \right) du \wedge dv$$

$$= \left( \frac{\partial (F_\varphi \cdot uv)}{\partial v} - \frac{\partial (F_v \sqrt{u^2+v^2})}{\partial \varphi} \right) \frac{e^v \wedge e}{uv\sqrt{u^2+v^2}}$$

$$+ \left( \frac{\partial (F_u \cdot \sqrt{u^2+v^2})}{\partial \varphi} - \frac{\partial (F_\varphi \cdot uv)}{\partial u} \right) \frac{e^\varphi \wedge e^u}{uv\sqrt{\ldots}} \left( \frac{\partial (F_v \sqrt{u^2+v^2})}{\partial u} - \frac{\partial (F_u \sqrt{\ldots}}{\partial v} \right) \frac{e^u \wedge e^v}{u^2+v^2}$$

$$\frac{1}{uv\sqrt{u^2+v^2}}\left(\frac{\partial F_\varphi}{\partial v}uv+uF_\varphi-\frac{\partial F_v}{\partial\varphi}\sqrt{u^2+v^2}\right)e^v\wedge e^\varphi +$$

$$+\frac{1}{uv\sqrt{u^2+v^2}}\left(\frac{\partial F_u}{\partial\varphi}\sqrt{u^2+v^2}-\frac{\partial F_\varphi}{\partial u}uv-vF_\varphi\right)e^\varphi\wedge e^u +$$

$$+\frac{1}{u^2+v^2}\left(\frac{\partial F_v}{\partial u}\sqrt{u^2+v^2}+\frac{u}{\sqrt{u^2+v^2}}F_v-\frac{\partial F_u}{\partial v}\sqrt{u^2+v^2}-\right.$$

$$\left.-F_u\frac{v}{\sqrt{u^2+v^2}}\right)e^u\wedge e^v =$$

$$=\left[\frac{1}{\sqrt{u^2+v^2}}\left(\frac{\partial F_\varphi}{\partial v}+\frac{1}{v}F_\varphi\right)-\frac{1}{uv}\frac{\partial F_v}{\partial\varphi}\right]e^v\wedge e^\varphi +$$

$$+\left[\frac{1}{uv}\frac{\partial F_u}{\partial\varphi}-\frac{1}{\sqrt{u^2+v^2}}\left(\frac{\partial F_\varphi}{\partial u}+\frac{1}{u}F_\varphi\right)\right]e^\varphi\wedge e^u +$$

$$+\left[\frac{1}{\sqrt{u^2+v^2}}\left(\frac{\partial F_v}{\partial u}-\frac{\partial F_u}{\partial v}\right)+\frac{uF_v-vF_u}{u^2+v^2}\right]e^u\wedge e^v$$

V hranatých závorkách složky rotace.
(Souhlasí s Michalovým výpočtem.)

# Gradient

$g = g(x, y, z)$, $\alpha : (u, v, w) \longrightarrow (x, y, z)$

$g \circ \alpha = g(x(u,v,w), y(u,v,w), z(u,v,w))$

$$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz = \omega^{(1)}_{\mathrm{grad}\, g}$$

současně platí

$$d(g \circ \alpha) = \frac{\partial (g \circ \alpha)}{\partial u} du + \frac{\partial (g \circ \alpha)}{\partial v} dv + \frac{\partial (g \circ \alpha)}{\partial w} dw$$

$$= \frac{\partial (g \circ \alpha)}{\partial u} \frac{1}{|f_u|} e^u + \frac{\partial (g \circ \alpha)}{\partial v} \frac{1}{|f_v|} e^v +$$

$$+ \frac{\partial (g \circ \alpha)}{\partial w} \frac{1}{|f_w|} e^w \Rightarrow$$

$$\mathrm{grad}(g \circ \alpha) = \left( \frac{1}{|f_u|} \frac{\partial (g \circ \alpha)}{\partial u}, \frac{1}{|f_v|} \frac{\partial (g \circ \alpha)}{\partial v}, \frac{1}{|f_w|} \frac{\partial (g \circ \alpha)}{\partial w} \right)$$

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Kontrola - sférické

$f_r \sim (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, $|f_r| = 1$

$f_\vartheta \sim (r\cos\vartheta\cos\varphi, r\cos\vartheta\sin\varphi, -\sin\vartheta)$, $|f_\vartheta| = r$

$f_\varphi \sim (-r\sin\vartheta\sin\varphi, r\sin\vartheta\cos\varphi, 0)$, $|f_\varphi| = r\sin\vartheta$

$$\operatorname{grad}(g\circ\alpha) = \left(\frac{\partial(g\circ\alpha)}{\partial r}, \frac{1}{r}\frac{\partial(g\circ\alpha)}{\partial\vartheta}, \frac{1}{r\sin\vartheta}\frac{\partial(g\circ\alpha)}{\partial\varphi}\right)$$

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Parabolické: $|f_u| = \sqrt{u^2+v^2}$, $|f_v| = \sqrt{u^2+v^2}$, $|f_w| = uv$

$$\operatorname{grad}(g\circ\alpha) = \left(\frac{1}{\sqrt{u^2+v^2}}\frac{\partial(g\circ\alpha)}{\partial u}, \frac{1}{\sqrt{u^2+v^2}}\frac{\partial(g\circ\alpha)}{\partial v}, \frac{1}{uv}\frac{\partial(g\circ\alpha)}{\partial\varphi}\right)$$

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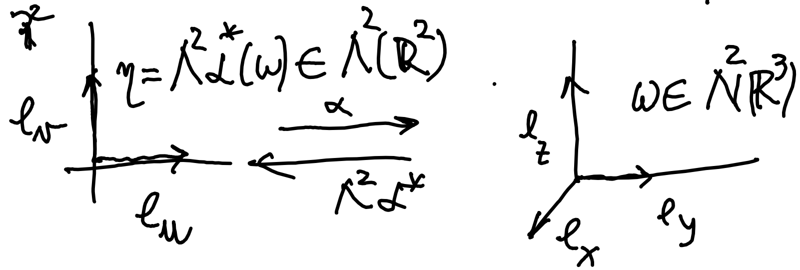
# Pullback

$\alpha : \mathbb{R}^2 \to \mathbb{R}^3 \qquad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{matrix} \dots f_u \\ \dots f_v \end{matrix}$




$f_u = \alpha(e_u)$, $f_v = \alpha(e_v)$ ... obrazy báze
(složky v řádcích matice)

$f_u = e_x - e_z$, $f_v = e_y$

Konstruujeme $\Lambda^2\alpha^*$: $\quad \eta = \Lambda^2 \alpha^* \omega \quad , \omega \in \Lambda^2(\mathbb{R}^3)$

$$\omega = \omega_x e^y \wedge e^z + \omega_y e^z \wedge e^x + \omega_z e^x \wedge e^y$$

$$\underbrace{\Lambda^2 \alpha^* \omega}_{\eta = \eta_0 e^u \wedge e^v} (e_u, e_v) = \omega_x e^y \wedge e^z (f_u, f_v) + \omega_y e^z \wedge e^x (f_u, f_v) + \omega_z e^x \wedge e^y (f_u, f_v)$$

$$= \omega_x (f_u^y f_v^z - f_u^z f_v^y) + \omega_y (f_u^z f_v^x - f_u^x f_v^z) + \omega_z (f_u^x f_v^y - f_u^y f_v^x) = -\omega_x + \omega_z \Rightarrow$$

$$\Rightarrow \eta_0 = \underbrace{\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}}_{A(\Lambda^2 \alpha^*)} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = -\omega_x + \omega_z$$

matice pullbacku $\Lambda^2 \alpha^*$ je $\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$