GRADIENT A DIVERGENCE
V BÁZI (ORTONORMÁLNÍ) VÁLCOVÝCH SOUŘADNIC

$(r, \varphi, z) \longleftrightarrow (x, y, z)$ $\quad x = r\cos\varphi,\ y = r\sin\varphi,\ z = z$

$\qquad r = \sqrt{x^2+y^2},\ \varphi = \operatorname{arctg}\frac{y}{x},\ z = z$

ONB $\begin{pmatrix} e_r \\ e_\varphi \\ e_z \end{pmatrix} = T \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$; $\quad T = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $T^{-1} = T^t$ (transp)

DUÁLNÍ $(e^r\ e^\varphi\ e^z) = (e^x\ e^y\ e^z)\,T^t$; $\quad (e^x\ e^y\ e^z) = (e^r\ e^\varphi\ e^z)\,T$

(to jsme si odvodili v pondělí) $\quad \underline{(e^x\ e^y\ e^z) = (dx\ dy\ dz)}$

$\omega^{(1)}_{\mathrm{grad} f} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$

df je objekt nezávislý na bázi, proto také $df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \varphi}d\varphi + \frac{\partial f}{\partial z}dz$

Stačí zjistit vztahy mezi bázi $(dr, d\varphi, dz)$ a (e^r, e^φ, e^z):

$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy = \cos\varphi\, e^x + \sin\varphi\, e^y = e^r$

$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = \ldots = -\frac{\sin\varphi}{r}dx + \frac{\cos\varphi}{r}dy = -\frac{\sin\varphi}{r}e^x + \frac{\cos\varphi}{r}e^y$

$\Rightarrow d\varphi = \frac{1}{r}e^\varphi$!!

$dz = e^z$

Pak $\quad df = \frac{\partial f}{\partial r}e^r + \frac{1}{r}\frac{\partial f}{\partial \varphi}e^\varphi + \frac{\partial f}{\partial z}e^z \Rightarrow \mathrm{grad} f(r, \varphi, z) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z}\right)$

v bázi $(e_r, e_\varphi, e_z$

$\omega^{(2)}_F = F_x\, dy\wedge dz + F_y\, dz\wedge dx + F_z\, dx\wedge dy = i_F(dx\wedge dy\wedge dz) = i_F\omega_0$

$d\omega^{(2)}_F = \omega^{(3)}_{\mathrm{div} F} = \mathrm{div}\, F\ \omega_0$

$\omega_0 = dx\wedge dy\wedge dz = e^x\wedge e^y\wedge e^z = e^r\wedge e^\varphi\wedge e^z = dr\wedge r\,d\varphi\wedge dz$

Ať $F \sim (F_r, F_\varphi, F_z)$, tj. $F = F_r e_r + F_\varphi e_\varphi + F_z e_z$

$i_F\omega_0 = F_r\, r\, d\varphi\wedge dz - F_\varphi\, dr\wedge dz + F_z\, r\, dr\wedge d\varphi$

Pozn. $\quad i_F e^\varphi = F_\varphi,\quad i_F(r\,d\varphi) = F_\varphi$

$\qquad i_F e^r = F_r,\quad i_F e^z = F_z$

./.

$$d\, i_F \omega_0 = d\left(F_r\, r\, d\varphi\wedge dz - F_\varphi\, dr\wedge dz + F_z\, r\, dr\wedge d\varphi\right) =$$

$$= dF_r \wedge r\, d\varphi\wedge dz + F_r\, dr\wedge d\varphi\wedge dz - dF_\varphi\wedge dr\wedge dz +$$

$$+ dF_z\wedge r\, dr\wedge d\varphi = r\frac{\partial F_r}{\partial r}\, dr\wedge d\varphi\wedge dz + F_r\, dr\wedge d\varphi\wedge dz -$$

$$- \frac{\partial F_\varphi}{\partial \varphi}\, d\varphi\wedge dr\wedge dz + \frac{\partial F_z}{\partial z}\, dz\wedge r\, dr\wedge dz =$$

$$= \left(\frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\varphi}{\partial \varphi} - \frac{\partial F_z}{\partial \varphi}\right) dr\wedge r\, d\varphi\wedge dz =$$

$$= \Big(\underbrace{\frac{\partial F_r}{\partial r} + \frac{1}{r}\frac{\partial F_\varphi}{\partial \varphi} + \frac{1}{r}F_r + \frac{\partial F_z}{\partial z}}_{\operatorname{div} F}\Big)\, e^r\wedge e^\varphi\wedge e^z$$

pro $F \sim \left(F_r(r,\varphi,z),\ F_\varphi(r,\varphi,z),\ F_z(r,\varphi,z)\right)$ v bázi (e_r, e_φ, e_z)

Prosím, raději zkontrolujte.