4. Meissner - Ochsenfeld effect

- current densities

coupling to EM fields

interaction Hamiltonian

$$\frac{1}{2m}\hat{p}^2 \rightarrow \frac{1}{2m}(\hat{p}-q\bar{A})^2 + q\phi \quad \rightarrow \quad H_{int} = \frac{q^2A^2}{2m} - \frac{q}{2m}(\hat{p}\cdot\bar{A} + \bar{A}\cdot\hat{p}) + q\phi$$

electrodynamics : density of work related to the change of EM fields

$$-\bar{j}\cdot\delta\bar{A} + \rho\,\delta\phi \quad \rightarrow \quad \bar{j}(\bar{r}) = -\frac{\delta H_{int}}{\delta\bar{A}(\bar{r})} \qquad \rho(\bar{r}) = \frac{\delta H_{int}}{\delta\phi(\bar{r})}$$

gauge transformation

$$\bar{E} = -\nabla\phi - \frac{\partial\bar{A}}{\partial t} \qquad \bar{A} \rightarrow \bar{A} + \nabla\chi$$

$$\bar{B} = \nabla\times\bar{A} \qquad \phi \rightarrow \phi - \frac{\partial\chi}{\partial t}$$

continuity equation

$$\nabla\cdot\bar{j} + \frac{\partial\rho}{\partial t} = 0$$

- many-electron operators

$$H_{int} = \underbrace{\sum_i \frac{e^2}{2m} \bar{A}^2(\bar{r}_i)}_{\text{diamagnetic}} + \underbrace{\frac{e}{2m} \sum_i \left[\bar{p}_i \cdot \bar{A}(\bar{r}_i) + \bar{A}(\bar{r}_i) \cdot \bar{p}_i\right]}_{\text{paramagnetic}} - e \sum_i \phi(\bar{r}_i)$$

→ current density

$$\bar{j}(\bar{r}) = -\frac{\delta H_{int}}{\delta \bar{A}(\bar{r})} = -\frac{e^2}{m} \sum_i \bar{A}(\bar{r}_i)\, \delta(\bar{r}-\bar{r}_i)$$

← diamagnetic $\bar{j}_d$

$$- \frac{e}{2m} \sum_i \left[\bar{p}_i\, \delta(\bar{r}-\bar{r}_i) + \delta(\bar{r}-\bar{r}_i)\, \bar{p}_i\right]$$

paramagnetic $\bar{j}_p$

→ charge density

$$\rho(\bar{r}) = \frac{\delta H_{int}}{\delta \phi(\bar{r})} = -e \sum_i \delta(\bar{r}-\bar{r}_i)$$

• second quantized form of the operators

$$H_{int} = \underbrace{\frac{e}{2m}\sum_i \left[\bar{p}_i\cdot\bar{A}(\bar{r}_i) + \bar{A}(\bar{r}_i)\cdot\bar{p}_i\right]}_{\text{paramagnetic}} + \ldots = \sum_i H_1(i) + \ldots$$

$$H_1 = \frac{e}{2m}\left[\bar{p}\cdot\bar{A}(\bar{r}) + \bar{A}(\bar{r})\cdot\bar{p}\right]$$

$$H_{int} = \sum_{k\sigma}\sum_{k'\sigma'} \langle k'\sigma'|H_1|k\sigma\rangle\, c^{+}_{k'\sigma'} c_{k\sigma} \qquad |k\sigma\rangle \to \frac{e^{i\bar{k}\cdot\bar{r}}}{\sqrt{V}} \times \text{spin part}$$

$$\langle k'\sigma'|H_1|k\sigma\rangle = \delta_{\sigma\sigma'}\frac{e}{2m}\int d^3\bar{r}\,\frac{e^{-i\bar{k}'\cdot\bar{r}}}{\sqrt{V}}\left(\frac{\hbar}{i}\nabla\cdot\underbrace{\bar{a}_q e^{i\bar{q}\cdot\bar{r}}}_{\bar{A}(\bar{r})} + \ldots\right)\frac{e^{i\bar{k}\cdot\bar{r}}}{\sqrt{V}}$$

$$= \delta_{\sigma\sigma'}\frac{e\hbar}{2m}\,\bar{a}_q\cdot\left(\bar{k}+\frac{\bar{q}}{2}\right)\delta_{k',k+q}$$

$$\rightarrow \quad H_{int} = \frac{e\hbar}{2m}\sum_{k\sigma}\left(\bar{k}+\cancel{\frac{\bar{q}}{2}}\right)\cdot\bar{a}_q\, c^{+}_{k+q\,\sigma} c_{k\sigma}$$

$$\nabla\cdot\bar{A} = 0$$
$$\bar{q}\cdot\bar{a}_q = 0$$

normalized paramag.
current density:

$$-\frac{e}{2m}\sum_i\left[\bar{p}_i\,\delta(\bar{r}-\bar{r}_i)+\delta(\bar{r}-\bar{r}_i)\,\bar{p}_i\right]$$

$$\bar{j}_p(\bar{q}) = \frac{1}{V}\int d^3\bar{r}\; e^{-i\bar{q}\cdot\bar{r}}\,\bar{j}_p(\bar{r})$$

$$= -\frac{e}{2m}\frac{1}{V}\sum_i\left(\bar{p}_i\, e^{-i\bar{q}\cdot\bar{r}_i} + e^{-i\bar{q}\cdot\bar{r}_i}\,\bar{p}_i\right) = \sum_i \bar{j}_1(i)$$

$$\rightarrow \quad \bar{j}_p(\bar{q}) = -\frac{e\hbar}{m}\frac{1}{V}\sum_{k\sigma}\left(\bar{k}-\frac{\bar{q}}{2}\right)c^{+}_{k-q\,\sigma}c_{k\sigma}$$

diamagnetic current density

...

$$\rightarrow \quad \bar{j}_d(\bar{q}) = -\frac{e^2}{m}\frac{1}{V}\,\bar{a}_q\sum_{k\sigma}c^{+}_{k\sigma}c_{k\sigma}$$

(only at $\bar{q}$ of the vector potential)

- perturbation theory to 1st order in $\bar{A}$

only paramagnetic part $\sim \bar{A}$

$$|\Psi_A\rangle \approx |\Psi_0\rangle - \sum_n \frac{\langle n|H_{int}|\Psi_0\rangle}{\underbrace{E_n - E_0}_{\text{excitation energy}}} |n\rangle$$

ground state for zero field

excited states

utilize the representation in terms of bogolons

$$c_{k\uparrow} = u_k^* b_{k\uparrow} + v_k b_{-k\downarrow}^+$$

$$c_{-k\downarrow}^+ = -v_k^* b_{k\uparrow} + u_k b_{-k\downarrow}^+$$

$|\Psi_0\rangle$ is bogolon vacuum

$$H_{BCS} = \sum_k E_k (b_{k\uparrow}^+ b_{k\uparrow} + b_{k\downarrow}^+ b_{k\downarrow})$$