4. Meissner-Ochsenfeld effect

Superconductor in a static magnetic field $\vec{A}(\vec{r}) = \vec{a}_q e^{i\vec{q}\cdot\vec{r}}$.

The additional term in the Hamiltonian $\hat{H}_{BCS}$ due to the presence of magnetic field is $\hat{H}_{int}$, where

$$\hat{H}_{int} = \frac{e}{2m} \sum_i \left[\underbrace{\hat{\vec{p}}_i \cdot \vec{A}(\vec{r}_i) + \vec{A}(\vec{r}_i) \cdot \hat{\vec{p}}_i}_{\text{paramagnetic}} \right]$$

The operator of paramagnetic current density in a superconductor $\hat{\vec{j}}_p(\vec{r})$ due to the presence of magnetic field is

$$\hat{\vec{j}}_p(\vec{r}) = -\frac{e}{2m} \sum_i \left[\delta(\vec{r}-\vec{r}_i)\, \hat{\vec{p}}_i + \hat{\vec{p}}_i\, \delta(\vec{r}-\vec{r}_i) \right]$$

4.1 Second Quantized form of $\hat{H}_{int}$, $\hat{\vec{j}}_p(\vec{q})$

$\hat{\vec{j}}_p(\vec{q})$ is Fourier transform of $\hat{\vec{j}}_p(\vec{r})$

$$\hat{H}_{int} = \sum_{k\sigma} \sum_{k'\sigma'} \langle k'\sigma' | \hat{H}_1 | k\sigma \rangle \hat{c}^{\dagger}_{k'\sigma'} \hat{c}_{k\sigma} \qquad ; \quad |k\sigma\rangle = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}} \chi_s(\sigma)$$

$$\hat{H}_1 = \frac{e}{2m} \left[\hat{\vec{p}} \cdot \vec{A} + \vec{A} \cdot \hat{\vec{p}} \right]$$

$$\hat{H}_{int} = \frac{e}{2m} \sum_{k\sigma} \sum_{k'\sigma'} \langle k'\sigma' | \hat{\vec{p}} \cdot \vec{A} + \vec{A} \cdot \hat{\vec{p}} | k\sigma \rangle \hat{c}^{\dagger}_{k'\sigma'} \hat{c}_{k\sigma}$$

• evaluating matrix element $\langle k'\sigma' | \hat{\vec{p}}\cdot\vec{A} + \vec{A}\cdot\hat{\vec{p}} | k\sigma\rangle$

– operator $\hat{\vec{p}}\cdot\vec{A} + \vec{A}\cdot\hat{\vec{p}}$ can be simplified like this:

$$[\hat{\vec{p}},\vec{A}] = 0 \quad \text{for} \quad \underbrace{\vec{\nabla}\cdot\vec{A} = 0}_{\text{Coulomb gauge}}$$

$$(\hat{\vec{p}}\cdot\vec{A} - \vec{A}\cdot\hat{\vec{p}})\psi(\vec{r}) = -i\hbar\vec{\nabla}\cdot(\vec{A}\psi) + i\hbar\vec{A}\cdot\vec{\nabla}\psi$$

using the vector identity $\vec{\nabla}\cdot(\vec{A}\psi) = (\vec{\nabla}\cdot\vec{A})\psi + \vec{A}\cdot(\vec{\nabla}\psi)$

$$\Rightarrow -i\hbar(\vec{\nabla}\cdot\vec{A})\psi - i\hbar\vec{A}\cdot(\vec{\nabla}\psi) + i\hbar\vec{A}\cdot(\vec{\nabla}\psi) = -i\hbar(\vec{\nabla}\cdot\vec{A})\psi$$

so $[\hat{\vec{p}},\vec{A}] = -i\hbar\vec{\nabla}\cdot\vec{A} = 0$ for $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow$

from $[\hat{\vec{p}},\vec{A}] = 0 \Rightarrow \hat{\vec{p}}\cdot\vec{A} = \vec{A}\cdot\hat{\vec{p}}$

$$\langle k'\sigma'|\hat{\vec{p}}\cdot\vec{A} + \vec{A}\cdot\hat{\vec{p}}|k\sigma\rangle = \langle k'\sigma'|2\vec{A}\cdot\hat{\vec{p}}|k\sigma\rangle$$

$$\langle k'\sigma'|2\vec{A}\cdot\hat{\vec{p}}|k\sigma\rangle = \frac{1}{V}\int\sum_s e^{-i\vec{k}'\cdot\vec{r}}\chi_c^*(\sigma')2\vec{a}_q e^{i\vec{q}\cdot\vec{r}}\cdot\hat{\vec{p}}\, e^{i\vec{k}\cdot\vec{r}}\chi_s(\sigma)\, d\vec{r}^3$$

$$= -\frac{i\hbar 2}{V}\underbrace{\sum_s \chi_c^*(\sigma')\chi_s(\sigma)}_{\delta_{\sigma\sigma'}}\int e^{-i\vec{k}'\cdot\vec{r}} e^{i\vec{q}\cdot\vec{r}}\vec{a}_q\cdot(i\vec{k})e^{i\vec{k}\cdot\vec{r}} d^3r$$

$$= -\frac{2i^2\hbar}{V}\delta_{\sigma\sigma'}\vec{k}\cdot\vec{a}_q\underbrace{\int e^{i(\vec{k}+\vec{q}-\vec{k}')\cdot\vec{r}}d\vec{r}^3}_{V\delta_{k+q,k'}} = \frac{\hbar 2}{V}\vec{k}\cdot\vec{a}_q V\delta_{k+q,k'}\delta_{\sigma\sigma'}$$

$$\hat{H}_{int} = \sum_{k'\sigma'}\sum_{k\sigma} \langle k'\sigma'|\hat{H}_1|k\sigma\rangle \hat{c}^{\dagger}_{k'\sigma'} \hat{c}_{k\sigma} = \frac{2e\hbar}{2m} \sum_{k'\sigma'}\sum_{k\sigma} \vec{k}\cdot\vec{a}_q\, \delta_{\sigma\sigma'}\, \delta_{k+q,k'}\, \hat{c}^{\dagger}_{k'\sigma'} \hat{c}_{k\sigma}$$

$$\boxed{= \frac{e\hbar}{m} \sum_{k\sigma} \vec{k}\cdot\vec{a}_q\, \hat{c}^{\dagger}_{k+q,\sigma} \hat{c}_{k\sigma}}$$

- paramagnetic current density:

- first the Fourier transform of $\hat{\vec{j}}_p(\vec{r})$

$$\hat{\vec{j}}_p(\vec{q}) = \int dr^3\, e^{-i\vec{q}\cdot\vec{r}}\, \hat{\vec{j}}_p(\vec{r}) = -\frac{e}{2m} \int \sum_i \left[\delta(\vec{r}-\vec{r}_i)\hat{\vec{p}}_i + \hat{\vec{p}}_i\,\delta(\vec{r}-\vec{r}_i)\right] e^{-i\vec{q}\cdot\vec{r}}\, d\vec{r}^3$$

$$= -\frac{e}{2m} \sum_i \int \left(\delta(\vec{r}-\vec{r}_i)\, e^{-i\vec{q}\cdot\vec{r}}\, \hat{\vec{p}}_i + \hat{\vec{p}}_i\, \delta(\vec{r}-\vec{r}_i)\, e^{-i\vec{q}\cdot\vec{r}}\right) dr^3 =$$

$$= -\frac{e}{2m} \sum_i \left[\hat{\vec{p}}_i e^{-i\vec{q}\cdot\vec{r}_i} + e^{-i\vec{q}\cdot\vec{r}_i}\hat{\vec{p}}_i\right] = \hat{\vec{j}}_p(q)$$

In 2nd Quantized form

$$\hat{\vec{j}}_p(q) = \sum_{k\sigma}\sum_{k'\sigma'} \langle k'\sigma'| -\frac{e}{2m}\left(\hat{\vec{p}}\, e^{-i\vec{q}\cdot\vec{r}} + e^{-i\vec{q}\cdot\vec{r}}\,\hat{\vec{p}}\right)|k\sigma\rangle\, \hat{c}^{\dagger}_{k'\sigma'} \hat{c}_{k\sigma}$$

- evaluating the matrix element $\langle k'\sigma'| \cdots |k\sigma\rangle$

$$\langle k'\sigma' | \dots | k\sigma \rangle = -\frac{e}{2m}\frac{1}{V}\sum_s \int e^{-i\vec{k}'\cdot\vec{r}}\chi_s^*(r')\left(\hat{\vec{p}}\,e^{-i\vec{q}\cdot\vec{r}} + e^{-i\vec{q}\cdot\vec{r}}\,\hat{\vec{p}}\right)e^{i\vec{k}\cdot\vec{r}}\chi_s(r)\,dr^3$$

$$= -\frac{e}{2m}\frac{1}{V}\underbrace{\sum_s \chi_s^*(r')\chi_s(\sigma)}_{\delta_{\sigma\sigma'}}\int dr^3\left[e^{-i\vec{k}'\cdot\vec{r}}\left(-i\hbar\vec{\nabla}e^{i(\vec{k}-\vec{q})\cdot\vec{r}} + e^{-i\vec{q}\cdot\vec{r}}(-i\hbar)\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}\right)\right]$$

$$= \frac{i\hbar e}{2mV}\delta_{\sigma\sigma'}\int\left(i(\vec{k}-\vec{q})e^{i(\vec{k}-\vec{q}-\vec{k}')\cdot\vec{r}} + i\vec{k}e^{i(\vec{k}-\vec{q}-\vec{k}')\cdot\vec{r}}\right)dr^3 =$$

$$= -\frac{\hbar e}{2mV}\delta_{\sigma\sigma'}\int(2\vec{k}-\vec{q})e^{i(\vec{k}-\vec{q}-\vec{k}')\cdot\vec{r}}\,dr^3$$

$$= -\frac{\hbar e}{2m}\frac{1}{V}\delta_{\sigma\sigma'}(2\vec{k}-\vec{q})\underbrace{\int e^{i(\vec{k}-\vec{q}-\vec{k}')\cdot\vec{r}}\,dr^3}_{V\delta_{k,q+k'}}$$

$$= -\frac{\hbar e}{m}\frac{1}{V}\delta_{\sigma\sigma'}\left(\vec{k}-\frac{\vec{q}}{2}\right)V\delta_{k,q+k'}$$

$$\hat{\vec{j}}_p(q) = \sum_{k\sigma}\sum_{k'\sigma'}\left(-\frac{\hbar e}{m}\delta_{\sigma\sigma'}(\vec{k}-\vec{q}/2)\right)\delta_{k,q+k'}\,\hat{c}^\dagger_{k'\sigma'}\hat{c}_{k\sigma} = \qquad \begin{matrix} k = q+k' \\ k' = k-q \end{matrix}$$

$$=$$

$$\Rightarrow \quad \Rightarrow \boxed{-\frac{e\hbar}{m}\sum_{k\sigma}\left(\vec{k}-\frac{\vec{q}}{2}\right)\hat{c}^\dagger_{k-q,\sigma}\hat{c}_{k\sigma} = \hat{\vec{j}}_p(q)}$$

4.2 Meissner effect

- Perturbation theory to the 1^{st} order. State vector $|\psi_A\rangle$ for a superconductor in a weak magnetic field $|\psi_A\rangle = |\psi_0\rangle - \sum_n \frac{\langle n|\hat{H}_{int}|\psi_0\rangle}{E_n - E_0}|n\rangle$

$|\psi_0\rangle$ is BCS state

$|\psi_0\rangle = \prod_k (u_k + v_k \hat{c}^{+}_{k\uparrow}\hat{c}_{-k\downarrow})|0\rangle$, where $|0\rangle$ is vacuum state

$|n\rangle$ = excited states

E_n = excitation energies
E_0

- computing the mean value of $\vec{\hat{j}}_P(q) = \langle\psi_A|\vec{\hat{j}}_P(q)|\psi_A\rangle$

$$\vec{j}_P(q) = \left(\langle\psi_0| - \sum_{n'} \frac{\langle\psi_0|\hat{H}_{int}|n'\rangle}{E_{n'} - E_0}\langle n'|\right)\vec{\hat{j}}_P(q)\left(|\psi_0\rangle - \sum_n \frac{\langle n|\hat{H}_{int}|\psi_0\rangle}{E_n - E_0}|n\rangle\right)$$

$$= \langle\psi_0|\vec{\hat{j}}_P(q)|\psi_0\rangle - \sum_n \frac{\langle n|\hat{H}_{int}|\psi_0\rangle}{E_n - E_0}\langle\psi_0|\vec{\hat{j}}_P(q)|n\rangle$$

$$- \sum_{n'} \frac{\langle\psi_0|\hat{H}_{int}|n'\rangle}{E_{n'} - E_0}\langle n'|\vec{\hat{j}}_P(q)|\psi_0\rangle + \sum_{nn'} \frac{\langle\psi_0|\hat{H}_{int}|n'\rangle\langle n|\hat{H}_{int}|\psi_0\rangle}{(E_{n'} - E_0)(E_n - E_0)}\langle n'|\vec{\hat{j}}_P(q)|n\rangle$$

• evaluating ~~the matrix elements~~ $\langle \psi_0 | \vec{j}_p(q) | \psi_0 \rangle$ and $\langle n | \hat{H}_{int} | \psi_0 \rangle$

$$\hat{H}_{int} = \frac{e\hbar}{m} \sum_{k\sigma} \vec{k} \cdot \vec{a}_q \hat{c}^{\dagger}_{k+q,\sigma} \hat{c}_{k\sigma} = \frac{e\hbar}{m} \sum_{k} \vec{k} \cdot \vec{a}_q \left(\hat{c}^{\dagger}_{k+q\uparrow} \hat{c}_{k\uparrow} - \hat{c}^{\dagger}_{-k\downarrow} \hat{c}_{-k-q\downarrow} \right)$$

$$\langle n | \hat{H}_{int} | \psi_0 \rangle = \langle n | \frac{e\hbar}{m} \sum_{k} \vec{k} \cdot \vec{a}_q \left(\hat{c}^{\dagger}_{k+q\uparrow} \hat{c}_{k\uparrow} - \hat{c}^{\dagger}_{-k\downarrow} \hat{c}_{-k-q\downarrow} \right) | \psi_0 \rangle$$

$\langle n | \hat{c}^{\dagger}_{k+q\uparrow} \hat{c}_{k\uparrow} | \psi_0 \rangle$; we utilize the representation in terms of bogolons

$$\hat{c}_{k\uparrow} = u_k^* \hat{b}_{k\uparrow} + v_k \hat{b}^{\dagger}_{-k\downarrow} \qquad ; \qquad \hat{c}^{\dagger}_{k\uparrow} = u_k \hat{b}^{\dagger}_{k\uparrow} + v_k \hat{b}_{-k\downarrow}$$

$$\hat{c}^{\dagger}_{-k\downarrow} = -v_k \hat{b}_{k\uparrow} + u_k \hat{b}^{\dagger}_{-k\downarrow} \qquad ; \qquad \hat{c}_{-k\downarrow} = u_k \hat{b}_{-k\downarrow} - v_k \hat{b}^{\dagger}_{k\uparrow}$$

$$\langle n | \left(u_{k+q} \hat{b}^{\dagger}_{k+q\uparrow} + v_{k+q} \hat{b}_{-k-q\downarrow} \right) \left(u_k \hat{b}_{k\uparrow} + v_k \hat{b}^{\dagger}_{-k\downarrow} \right) | \psi_0 \rangle$$

$$= \langle n | u_{k+q} v_k \hat{b}^{\dagger}_{k+q\uparrow} \hat{b}^{\dagger}_{-k\downarrow} | \psi_0 \rangle \quad ; \text{because} \quad \hat{b}_{k\uparrow} | \psi_0 \rangle = 0$$
$$\hat{b}_{-k-q\downarrow} | \psi_0 \rangle = 0$$

$|\psi_0\rangle$ is bogolon vacuum

$$\langle n | \hat{c}^{\dagger}_{-k\downarrow} \hat{c}_{-k-q\downarrow} | \psi_0 \rangle = \langle n | \left(-v_k \hat{b}_{k\uparrow} + u_k \hat{b}^{\dagger}_{-k\downarrow} \right) \left(u_{k+q} \hat{b}_{-k-q\downarrow} - v_{k+q} \hat{b}^{\dagger}_{k+q\uparrow} \right) | \psi_0 \rangle$$

$$= \langle n | - u_k v_{k+q} \hat{b}^{\dagger}_{-k\downarrow} \hat{b}^{\dagger}_{k+q\uparrow} | \psi_0 \rangle \quad , \text{because} \quad \hat{b}_{-k-q\downarrow} | \psi_0 \rangle = 0$$
$$\hat{b}_{k\uparrow} | \psi_0 \rangle = 0$$

$$= \langle n | u_k v_{k+q} \hat{b}^{\dagger}_{k+q\uparrow} \hat{b}^{\dagger}_{-k\downarrow} | \psi_0 \rangle$$

using anticommutation relation $\{ \hat{b}^{\dagger}_{k\sigma}, \hat{b}^{\dagger}_{k'\sigma'} \} = \delta_{kk'} \delta_{\sigma\sigma'}$

$$\langle n|\hat{H}_{int}|\psi_0\rangle = \frac{e\hbar}{m}\sum_k \vec{k}\cdot\vec{a}_q \left(u_{k+q}v_k - u_k v_{k+q}\right)\langle n|\hat{b}^\dagger_{k+q\uparrow}\hat{b}^\dagger_{-k\downarrow}|\psi_0\rangle$$

For $\vec{q}\to 0$, we have $\quad u_{k+q}v_k - u_k v_{k+q} = u_k v_k - u_k v_k = 0$

and $\quad \langle n|\hat{H}_{int}|\psi_0\rangle \to 0$

- $\langle\psi_0|\hat{\vec{j}}_p(q)|\psi_0\rangle$:

$$\hat{\vec{j}}_p(q) = -\frac{e\hbar}{m}\sum_{k\sigma}\left(\vec{k}-\frac{\vec{q}}{2}\right)\hat{c}^\dagger_{k-q,\sigma}\hat{c}_{k\sigma} = -\frac{e\hbar}{m}\sum_k\left(\vec{k}-\frac{\vec{q}}{2}\right)\left[\hat{c}^\dagger_{k-q\uparrow}\hat{c}_{k\uparrow} - \hat{c}^\dagger_{-k\downarrow}\hat{c}_{-k+q\downarrow}\right]$$

$$\langle\psi_0|\hat{c}^\dagger_{k-q\uparrow}\hat{c}_{k\uparrow}|\psi_0\rangle = \langle\psi_0|\left(u_{k-q}\hat{b}^\dagger_{k-q\uparrow} + v_k\hat{b}_{-k+q\downarrow}\right)\left(u_k\hat{b}_{k\uparrow} + v_k\hat{b}^\dagger_{-k\downarrow}\right)|\psi_0\rangle$$

$$= \langle\psi_0|\, u_{k-q}v_k\,\hat{b}^\dagger_{k-q\uparrow}\hat{b}^\dagger_{-k\downarrow}|\psi_0\rangle \quad ; \quad \hat{b}_{-k+q\uparrow}|\psi_0\rangle \quad \hat{b}_{-k+q\downarrow}\hat{b}^\dagger_{-k\downarrow}|\psi_0\rangle = 0$$

$$\hat{b}_{k\uparrow}|\psi_0\rangle$$

$$\langle\psi_0|\hat{c}^\dagger_{-k\downarrow}\hat{c}_{-k+q\downarrow}|\psi_0\rangle = \langle\psi_0|\left(u_k\hat{b}^\dagger_{-k\downarrow} - v_k\hat{b}_{k\uparrow}\right)\left(u_{k-q}\hat{b}_{-k+q\downarrow} - v_{k-q}\hat{b}^\dagger_{k-q\uparrow}\right)|\psi_0\rangle$$

$$= \langle\psi_0| - u_k v_{k-q}\hat{b}^\dagger_{-k\downarrow}\hat{b}^\dagger_{k-q\uparrow}|\psi_0\rangle \quad ; \quad \hat{b}_{-k+q\downarrow}|\psi_0\rangle$$

$$= \langle\psi_0|\, u_k v_{k-q}\hat{b}^\dagger_{k-q\uparrow}\hat{b}^\dagger_{-k\downarrow}|\psi_0\rangle \qquad \hat{b}_{k\uparrow}\hat{b}^\dagger_{k-q\uparrow}|\psi_0\rangle$$

$$\langle\psi_0|\hat{\vec{j}}_p(q)|\psi_0\rangle = -\frac{e\hbar}{m}\sum_k\left(\vec{k}-\frac{\vec{q}}{2}\right)\left(u_{k-q}v_k - u_k v_{k-q}\right)\langle\psi_0|\hat{b}^\dagger_{k-q\uparrow}\hat{b}^\dagger_{-k\downarrow}|\psi_0\rangle$$

for $\vec{q}\to 0 \Rightarrow u_{k-q}v_k - u_k v_{k-q} \to u_k v_k - u_k v_k = 0$

$$\langle\psi_0|\hat{\vec{j}}_p|\psi_0\rangle \to 0$$

$\Rightarrow \langle \Psi_n | \hat{\vec{j}}_P \Psi_n \rangle = 0$ for $\vec{q} \to 0$

Paramagnetic current density approaches 0 for $\vec{q} \to 0$, leaving only diamagnetic part $\vec{j}_d = -\frac{ne^2}{m}\vec{A}$ (London equation)

$\vec{j}_P = 0$, $\vec{j}_d \to$ London equation and consequently, the Meissner effect