2. domácí úloha

KONZULTACE 9/4

① Rozhodněte, zda jde o kvadr. formy. Pokud ano, napište příslušnou sym. bilin. formu

(1) $G : \mathbb{R}_3[x] \to \mathbb{R}$

$$G(p) = p'(4) \int_{-1}^{1} p(x)\,dx$$

$f(p) = p'(4)$ lin. forma

$g(p) = \int_{-1}^{1} p(x)\,dx$ lin. forma

$$G(p) = f(p) \cdot g(p)$$

F příslušná sym. bilin. forma

$$H(p,q) = \underline{f(p)}\ \underline{g(q)} = p'(4) \int_{-1}^{1} q(x)\,dx$$

bilin. forma, ale není symetrická!

$$F(p,q) = \frac{1}{2} p'(4) \int_{-1}^{1} q(x)\,dx + \frac{1}{2} q'(4) \int_{-1}^{1} p(x)\,dx$$

(2) $H : Mat_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ $\quad X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$

$$H(X) = \det X = x_1 x_4 - x_2 x_3$$

$$H(X,Y) = x_1 y_4 - x_2 y_3$$ bilin. forma ale ne symetrická

$$F(X,Y) = \frac{1}{2} x_1 y_4 + \frac{1}{2} x_4 y_1 - \frac{1}{2} x_2 y_3 - \frac{1}{2} x_3 y_2$$

příslušná sym. bilin. forma

(3) $T : Mat_{3\times 3}(\mathbb{R}) \longrightarrow \mathbb{R}$

$$T(X) = X_{11} + X_{22} + X_{33}, \quad X = (X_{ij})$$

Q je kvadr. forma

$$\underline{Q(u)} = F(u,u) \qquad F \text{ bilin. sym.}$$

$t \in \mathbb{R}$

$$\underline{Q(tu)} = F(tu,tu) = t^2 F(u,u) = \underline{t^2 Q(u)}$$

Tato vlastnost pro T neplatí

$$T(aX) = aX_{11} + aX_{22} + aX_{33}$$
$$= a\,T(X) \neq a^2 T(X)$$

$T(X) \neq 0$ a $a \neq 1$.

2. úloha Dokažte

$$(x_1+2x_2+3x_3)^2 \le 14\,(x_1^2+x_2^2+x_3^2)$$

a zjistěte, kdy platí rovnost.

$$|\langle u,v\rangle| \le \|u\|\,\|v\|$$

a rovnost nastane právě když u a v jsou lin. závislé.

$\mathbb{R}^3$ $\langle x,y\rangle = x_1y_1+x_2y_2+x_3y_3$

$$|\langle x,(1,2,3)\rangle| = |x_1+2x_2+3x_3|$$

$$|x_1+2x_2+3x_3| \le \|(1,2,3)\|\,\|(x_1,x_2,x_3)\|$$

$$(x_1+2x_2+3x_3)^2 \le \underbrace{(1^2+2^2+3^2)}_{=14}(x_1^2+x_2^2+x_3^2)$$

Rovnost nastane pro

$$x = c\,(1,2,3)$$

$$x_1 = c,\quad x_2 = 2c,\quad x_3 = 3c$$

$c \in \mathbb{R}$.

(3) Kvadr. forma $q : \mathbb{R}^2 \to \mathbb{R}$ v souřadnicích x_1, x_2 báze $\alpha = ((0,1),(1,2))$ má vyjádření

$$q(u) = x_1^2 - 3x_1x_2 .$$

Najděte její vyjádření v souřadnicích y_1, y_2 stand. báze $\varepsilon = ((1,0),(0,1))$.

f příslušná sym. bilin. forma je

$$f(x,z) = x_1z_1 - \frac{3}{2}x_1z_2 - \frac{3}{2}x_2z_1$$

Matice f v bázi α je A

$$A = \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 0 \end{pmatrix}$$

B je matice f v bázi $\varepsilon = ((1,0),(0,1))$

$$B = (id)_{\alpha,\varepsilon}^T \, A \, (id)_{\alpha,\varepsilon}$$

$$(id)_{\alpha,\varepsilon} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \alpha = \begin{matrix} 0 & 1 \\ 1 & 2 \end{matrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-2)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{7}{2} & 1 \\ \frac{6}{2} & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 10 & -\frac{7}{2} \\ -\frac{7}{2} & 1 \end{pmatrix}$$

$$f(y,u) = 10y_1u_1 - \frac{7}{2}y_1u_2 - \frac{7}{2}y_2u_1 + y_2u_2$$

$$\boxed{q(y) = 10y_1^2 - 7y_1y_2 + y_2^2}$$

I trochu jinak

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (u)_\alpha = (id)_{\alpha,\varepsilon}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (id)_{\alpha,\varepsilon}(u)_\varepsilon$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$x_1 = -2y_1 + y_2 \qquad x_2 = y_1$$

Toto vyjádření dosadíme do kvadr. formy

$$g(x) = x_1^2 - 3x_1x_2$$

$$= (-2y_1+y_2)^2 - 3(-2y_1+y_2)y_1$$

$$= 4y_1^2 - 4y_1y_2 + y_2^2$$

$$+ 6y_1^2 - 3y_2y_1$$

$$= 10y_1^2 - 7y_1y_2 + y_2^2$$

4. příklad

$\mathbb{R}^4$

$$2x_1x_2 + 2x_1x_3 + 4x_1x_4 - x_2^2 + 3x_3^2 + 4x_3x_4 + 4x_4^2$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 2 & e_1 \\ 1 & -1 & 0 & 0 & e_2 \\ 1 & 0 & 3 & 2 & e_3 \\ 2 & 0 & 2 & 4 & e_4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & e_2 \\ 0 & 1 & 1 & 2 & e_1 \\ 1 & 0 & 3 & 2 & e_3 \\ 2 & 0 & 2 & 4 & e_4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & e_2 \\ 1 & 0 & 1 & 2 & e_1 \\ 0 & 1 & 3 & 2 & e_3 \\ 0 & 2 & 2 & 4 & e_4 \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & e_2 \\ 0 & 1 & 1 & 2 & e_1+e_2 \\ 0 & 1 & 3 & 2 & e_3 \\ 0 & 2 & 2 & 4 & e_4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & e_2 \\ 0 & 1 & 1 & 2 & e_1+e_2 \\ 0 & 1 & 3 & 2 & e_3 \\ 0 & 2 & 2 & 4 & e_4 \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & e_2 \\ 0 & 1 & 1 & 2 & e_1+e_2 \\ 0 & 0 & 2 & 0 & -e_1-e_2+e_3 \\ 0 & 0 & 0 & 0 & -2e_1-2e_2+e_4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|l} -1 & 0 & 0 & 0 & e_2 = \underline{(0,1,0,0)} \\ 0 & 1 & 0 & 0 & e_1+e_2 = \underline{(1,1,0,0)} \\ 0 & 0 & 2 & 0 & -e_1-e_2+e_3 = \underline{(-1,-1,1,0)} \\ 0 & 0 & 0 & 0 & -2e_1-2e_2+e_4 = \underline{(-2,-2,0,1)} \end{array}\right)$$

$$-y_1^2+y_2^2+2y_3^2$$

signatura $\quad s_+ = 2$

$s_- = 1$

$s_0 = \underline{1}$

3. domácí úloha

(1) G S ort. proces

$$u_1 = (1,-1,1,1)$$
$$u_2 = (2,3,1,0) \implies v_1, v_2, v_3 \text{ pomocí GS}$$
$$u_3 = (1,2,2,1)$$

$$v_1 = u_1 = (1,-1,1,1)$$

$$v_2 = u_2 - a v_1 \quad \perp v_1$$

$$0 = \langle v_1, v_2 \rangle = \langle u_2, v_1 \rangle - a \langle v_1, v_1 \rangle$$

$$a = \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{0}{\|v_1\|} = 0$$

$$v_2 = (2,3,1,0)$$

$$v_3 = u_3 - b v_1 - c v_2 \quad / \perp v_1, v_2$$

$$b = \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{2}{4} = \frac{1}{2}$$

$$c = \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} = \frac{10}{14} = \frac{5}{7}$$

$$v_3 = (1,2,2,1) - \frac{1}{2}(1,-1,1,-1) - \frac{5}{7}(2,3,1,0)$$
$$= \frac{1}{7}(-3,-1,9,7)$$

2. příklad

Spočítejte kolmou projekci vektoru $(1,1,-3,9) = u$ do $[v_1 = (-2,0,4,1),\ v_2 = (3,-1,-2,1)]$

$$Pu = \underline{a v_1 + b v_2}$$

$$u - Pu \perp v_1, v_2$$

$$\langle Pu, v_1 \rangle = \langle u, v_1 \rangle$$
$$\langle Pu, v_2 \rangle = \langle u, v_2 \rangle$$
$$a\langle v_1, v_1 \rangle + b\langle v_2, v_1 \rangle = \langle u, v_1 \rangle$$
$$a\langle v_1, v_2 \rangle + b\langle v_2, v_2 \rangle = \langle u, v_2 \rangle$$

$$21a - 13b = -5$$
$$-13a + 15b = 17$$

$$\begin{array}{cc} b & a \end{array}$$
$$\left(\begin{array}{cc|c} -13 & 21 & -5 \\ 15 & -13 & 17 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 8 & 12 \\ 15 & -13 & 17 \end{array}\right)$$

$$\begin{array}{cc} 2 & 1 \end{array}$$
$$\sim \left(\begin{array}{cc|c} 1 & 4 & 6 \end{array}\right)$$

$b = 2$
$a = 1$

$$Pu = 1 \cdot (-2,0,4,1) + 2(3,-1,-2,1) = \boxed{(4,-2,0,3)}$$

3. příklad $\quad P: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ je kolmá projekce na přímku

$$p: [0,0,0] + t(1,0,-1)$$

$$Px = A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad s_i(A) = P(e_i)$$

$$\begin{array}{ccc|ccc} & u & & & Pu & \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

$$A^T$$

$$P(x) = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

4. příklad vzdálenost $\quad -2+6-3-1=0$

$A=[3,-2,-1]$ od roviny

$\rho: \underline{2}x-6y+3z-1=0$

$B\in\rho \qquad Z(\rho)^{\perp}=[(2,-6,3)]$

$\text{dist}(A,\rho)=\|P_{Z(\rho)^{\perp}}(A-B)\|$

$B=[-1,-1,-1]\in\rho$

$A-B=(4,-1,0)$

$P_{Z(\rho)^{\perp}}(A-B)=a(2,-6,3)$

$(A-B)-P_{Z(\rho)^{\perp}}(A-B)\perp(2,-6,3)$

$$a=\frac{\langle \overset{(4,-1,0)}{A-B},(2,-6,3)\rangle}{\|(2,-6,3)\|^2}$$

$4+36+9$

$$=\frac{14}{49}=\frac{2}{7}$$

$\underline{\underline{\text{dist}}}=\frac{2}{7}\|(2,-6,3)\|=\frac{2}{7}\cdot 7=\underline{\underline{2}}$