Bernsteinove bázové polynómy:

$n \in \mathbb{N}$, $k \in \{0, \dots, n\}$, $x \in [0,1]$

$$b_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

nech f je reálna funkcia definovaná na $[0,1]$

Bernsteinov polynóm stupňa n funkcie f:

$$B_{f,n}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{k,n}(x)$$

Weierstrassova aproximačná veta: Ak $f \in C[a,b]$, potom pre ľubovoľné $\varepsilon > 0$ existuje polynóm p taký, že pre každé $x \in [a,b]$ platí $|f(x) - p(x)| < \varepsilon$.

(akúkoľvek spojitú funkciu vieme na kompaktnom intervale <u>rovnomerne</u> s ľubovoľnou presnosťou aproximovať polynómom.)

Dôkaz.: Tvrdenie dokážeme pre $[a,b] = [0,1]$. Pre všeobecný interval $[a,b]$ sa použije jednoduchá lineárna transformácia.

X_n náhodné veličiny, $X_n \sim Bi(n,x)$, $n \in \mathbb{N}$, $x \in [0,1]$

$$p_{X_n}(k) = P(X_n = k) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k}, & k \in \{0, \dots, n\} \\ 0, & \text{inak} \end{cases}$$

$E X_n = nx$

$D X_n = nx(1-x)$

pripomeňme Čebyševovu nerovnosť:

$$P(|X - EX| \geq \varepsilon) \leq \frac{DX}{\varepsilon^2}$$

nech $\varepsilon > 0$ je ľubovoľné a zvolíme ľubovoľne $0 < \varepsilon_0 < \varepsilon$.
(čiže o kúsok menšie kladné číslo)

f je spojitá na kompaktnom intervale $[0,1]$ $\Rightarrow$
$\Rightarrow$ f je rovnomerne spojitá $\Rightarrow$ pre ε_0 existuje $\delta > 0$
taká, že $\forall\, y, z \in [0,1]$ také, že $|y-z| < \delta$, platí
$$|f(y) - f(z)| < \varepsilon_0$$

podľa Čebyševovej nerovnosti platí:

$$P(|X_n - nx| \geq n\delta) \leq \frac{DX_n}{(n\delta)^2}$$

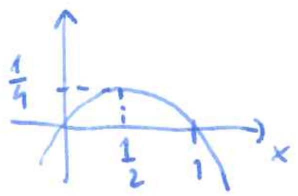
$$\parallel \qquad\qquad\qquad \parallel$$

$$P\left(\left|\frac{X_n}{n} - x\right| \geq \delta\right) \leq \frac{n\,x(1-x)}{n^2\delta^2} = x(1-x)\cdot\frac{1}{n\delta^2}$$

(odbočka: hľadáme maximum funkcie $g(x) = x(1-x)$ na intervale $[0,1]$

$g(x) = x(1-x) = -x^2 + x$, $g'(x) = -2x + 1 = 0 \Leftrightarrow x = \frac{1}{2}$

$g\left(\frac{1}{2}\right) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$



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$$\Rightarrow \quad P\left(\left|\frac{X_n}{n} - x\right| \geq \delta\right) \leq \frac{1}{4n\delta^2}$$

f spojitá $\Rightarrow$ f borelovsky merateľná $\wedge$ obor hodnôt $\frac{X_n}{n}$ je leží v $[0,1]$ $\Rightarrow$ $f\left(\frac{X_n}{n}\right)$ je náhodná veličina

keďže
$$\left\{\omega\in\Omega : \left|\frac{X_n(\omega)}{n}-x\right|<\delta\right\}$$
$$\subseteq\left\{\omega\in\Omega : \left|f\left(\frac{X_n(\omega)}{n}\right)-f(x)\right|<\varepsilon_0\right\}$$

platí
$$P\left(\left|\frac{X_n}{n}-x\right|<\delta\right)\leq P\left(\left|f\left(\frac{X_n}{n}\right)-f(x)\right|<\varepsilon_0\right)$$
$$-P\left(\left|\frac{X_n}{n}-x\right|<\delta\right)\geq -P\left(\left|f\left(\frac{X_n}{n}\right)-f(x)\right|<\varepsilon_0\right)$$
$$1-P\left(\left|\frac{X_n}{n}-x\right|<\delta\right)\geq 1-P\left(\left|f\left(\frac{X_n}{n}\right)-f(x)\right|<\varepsilon_0\right)$$
$$P\left(\left|\frac{X_n}{n}-x\right|\geq\delta\right)\geq P\left(\left|f\left(\frac{X_n}{n}\right)-f(x)\right|\geq\varepsilon_0\right)$$

čo znamená, že
$$P\left(\left|f\left(\frac{X_n}{n}\right)-f(x)\right|\geq\varepsilon_0\right)\leq\frac{1}{4n\delta^2}$$

teraz odhadneme strednú hodnotu n. v. $\left|f\left(\frac{X_n}{n}\right)-f(x)\right|$:

$$E\left|f\left(\frac{X_n}{n}\right)-f(x)\right| = \int\limits_{\left|f\left(\frac{X_n}{n}\right)-f(x)\right|<\varepsilon_0}\left|f\left(\frac{X_n(\omega)}{n}\right)-f(x)\right|dP(\omega)\ +$$
$$+\int\limits_{\left|f\left(\frac{X_n}{n}\right)-f(x)\right|\geq\varepsilon_0}\left|f\left(\frac{X_n(\omega)}{n}\right)-f(x)\right|dP(\omega)$$
$$\leq\int\limits_{\left|f\left(\frac{X_n}{n}\right)-f(x)\right|<\varepsilon_0}\varepsilon_0\,dP(\omega)+\int\limits_{\left|f\left(\frac{X_n}{n}\right)-f(x)\right|\geq\varepsilon_0}2M\,dP(\omega)$$

$\left(M=\max\limits_{y\in[0,1]}|f(y)|\right)$

$$\leq \int_{\Omega} \varepsilon_0 \, dP(\omega) + 2M \cdot \int_{\left|f\left(\frac{X_n}{n}\right) - f(x)\right| \geq \varepsilon_0} 1 \, dP(\omega)$$

$$= \varepsilon_0 + 2M \cdot P\left(\left|f\left(\frac{X_n}{n}\right) - f(x)\right| \geq \varepsilon_0\right) \leq \varepsilon_0 + 2M \cdot \frac{1}{4n\delta^2}$$

$$\leq \varepsilon$$

$\downarrow$

(pre dostatočne veľké n, lebo

$2M \cdot \frac{1}{4n\delta^2} \to 0$ pre $n \to \infty$)

z vlastnosti strednej hodnoty vyplýva

$$\left| E\left(f\left(\frac{X_n}{n}\right) - f(x)\right) \right| \leq E\left| f\left(\frac{X_n}{n}\right) - f(x) \right| \leq \varepsilon$$

$$\left| E\left(f\left(\frac{X_n}{n}\right)\right) - f(x) \right|$$

$$\left| \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k} - f(x) \right|$$

$$\left| B_{f,n}(x) - f(x) \right|$$

toto je to, čo sme chceli

príklad. Určte Bernsteinov polynóm stupňa 4 pre funkciu $f(x) = x^2$. $\quad n=4$

1. spôsob: $\cancel{b_{0,n}(x) = \binom{n}{0} x^0 (1-x)^n}$

$$b_{0,4}(x) = \binom{4}{0} x^0 (1-x)^4 = (1-x)^4$$
$$= x^4 - 4x^3 + 6x^2 - 4x + 1$$

$$b_{1,4}(x) = \binom{4}{1} x^1 (1-x)^3 = 4x(1-x)^3$$
$$= -4x^4 + 12x^3 - 12x^2 + 4x$$

$$b_{2,4}(x) = \binom{4}{2} x^2 (1-x)^2 = 6x^2(1-2x+x^2)$$
$$= 6x^4 - 12x^3 + 6x^2$$

$$b_{3,4}(x) = \binom{4}{3} x^3 (1-x)^1 = 4x^3(1-x) = -4x^4 + 4x^3$$

$$b_{4,4}(x) = \binom{4}{4} x^4 (1-x)^0 = x^4$$

$$B_{f,4}(x) = \sum_{k=0}^{4} f\left(\frac{k}{4}\right) b_{k,4}(x) = 0 \cdot (x^4 - 4x^3 + 6x^2 - 4x + 1)$$
$$+ \left(\frac{1}{4}\right)^2 \cdot (-4x^4 + 12x^3 - 12x^2 + 4x) + \left(\frac{1}{2}\right)^2 \cdot (6x^4 - 12x^3 + 6x^2)$$
$$+ \left(\frac{3}{4}\right)^2 \cdot (-4x^4 + 4x^3) + 1 \cdot x^4$$

$$= \frac{-1+6-9+4}{4} x^4 + \frac{3-12+9}{4} x^3 + \frac{-3+6}{4} x^2 + \frac{1}{4} x$$

$$= \frac{3}{4} x^2 + \frac{1}{4} x = x^2 + \frac{x - x^2}{4}$$

2. spôsob: $\quad f_i^{(0)} = f\left(\frac{i}{n}\right), \quad i = 0, \dots, n$

$$f_i^{(j)} = f_i^{(j-1)} \cdot (1-x) + f_{i+1}^{(j-1)} x, \quad i = 0, \dots, n-j$$
$$j = 1, \dots, n$$

i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$
0	0	$0\cdot(1-x)+\frac{1}{16}x = \frac{x}{16}$	$\frac{x}{16}(1-x)+\left(\frac{3}{16}x+\frac{1}{16}\right)x = \frac{1}{8}x^2+\frac{1}{8}x$
1	$\frac{1}{16}$	$\frac{1}{16}(1-x)+\frac{1}{4}x = \frac{3}{16}x+\frac{1}{16}$	$\left(\frac{3}{16}x+\frac{1}{16}\right)(1-x)+\left(\frac{5}{16}x+\frac{1}{4}\right)x = \frac{1}{8}x^2+\frac{3}{8}x+\frac{1}{16}$
2	$\frac{1}{4}$	$\frac{1}{4}(1-x)+\frac{9}{16}x = \frac{5}{16}x+\frac{1}{4}$	$\left(\frac{5}{16}x+\frac{1}{4}\right)(1-x)+\left(\frac{7}{16}x+\frac{9}{16}\right)x = \frac{1}{8}x^2+\frac{5}{8}x+\frac{1}{4}$
3	$\frac{9}{16}$	$\frac{9}{16}(1-x)+1\cdot x = \frac{7}{16}x+\frac{9}{16}$	
4	1		

$f_i^{(3)}$
$\left(\frac{1}{8}x^2+\frac{1}{8}x\right)(1-x)+\left(\frac{1}{8}x^2+\frac{3}{8}x+\frac{1}{16}\right)x = \frac{3}{8}x^2+\frac{3}{16}x$
$\left(\frac{1}{8}x^2+\frac{3}{8}x+\frac{1}{16}\right)(1-x)+\left(\frac{1}{8}x^2+\frac{5}{8}x+\frac{1}{4}\right)x = \frac{3}{8}x^2+\frac{9}{16}x+\frac{1}{16}$

$f_i^{(4)}$
$\left(\frac{3}{8}x^2+\frac{3}{16}x\right)(1-x)+\left(\frac{3}{8}x^2+\frac{9}{16}x+\frac{1}{16}\right)x = \frac{3}{4}x^2+\frac{1}{4}x$