numerická optimalizácia

príklad. Metódou

a) rozpolovania intervalu

b) zlatého rezu

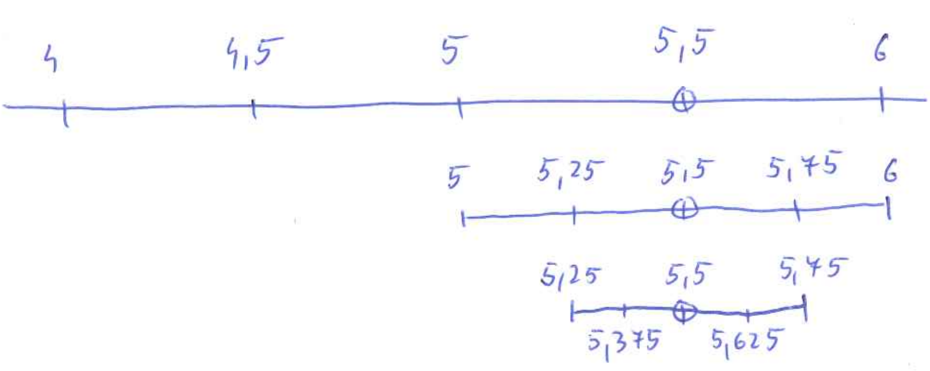
c) kvadratickej interpolácie

d) Newtonovou

odhadnite minimum funkcie $e^x \sin x$ na intervale $[4,6)$.
Spočítajte aspoň 3 iterácie.

a) v každom kroku rozdelíme aktuálny interval $[a,b]$ na 4 ~~rovné~~ podintervaly $[x_0,x_1], [x_1,x_2], [x_2,x_3], [x_3,x_4]$ rovnakej dĺžky. Nájdeme minimum z $f(x_1), f(x_2), f(x_3)$ $\leadsto$ označme $f(x_k)$ a pokračujeme s intervalom $[x_{k-1}, x_{k+1}]$.

krok						
1	x_i	4	4,5	5	5,5	6
	$f(x_i)$	−41,320	−87,994	−142,317	−172,640	−112,724
2	x_i	5	5,25	5,5	5,75	6
	$f(x_i)$	−142,317	−163,684	−172,640	−159,697	−112,724
3	x_i	5,25	5,375	5,5	5,625	5,75
	$f(x_i)$	−163,684	−170,245	−172,640	−169,603	−159,697

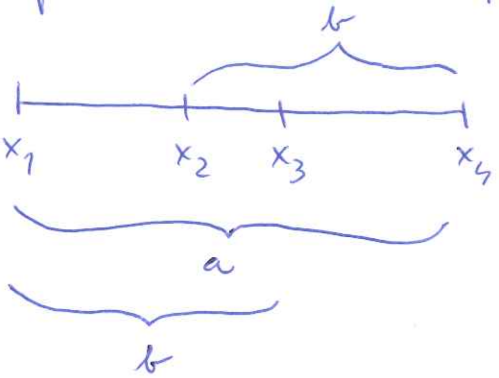


b) V aktuálnom intervale $[a, b]$ zvolíme body $x_1 = a$, x_2, x_3, $x_4 = b$ tak, aby platilo

$$\frac{x_4 - x_1}{x_3 - x_1} = \frac{x_4 - x_1}{x_4 - x_2} = \varphi = \frac{1+\sqrt{5}}{2}$$ (zlatý rez → φ je riešením rovnice $x^2 = x + 1$)

Nájdeme minimum z $f(x_2)$, $f(x_3)$ ⇝ označíme $f(x_k)$. V intervale $[x_{k-1}, x_k]$ zvolíme ďalšiu štvoricu bodov rovnakým spôsobom (stačí dopočítať len 1 bod).

Prečo:



$$\frac{a}{b} = \varphi, \quad \varphi^2 = \varphi + 1$$

$$\frac{a - b + b}{b} = \varphi$$

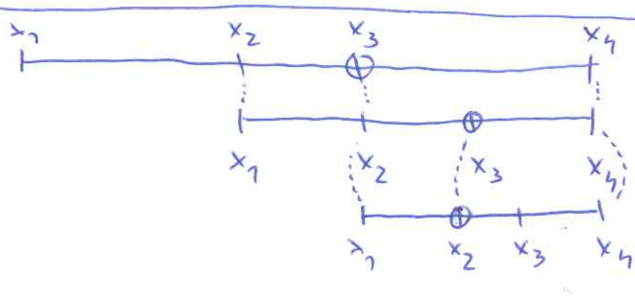
$$\frac{a-b}{b} + 1 = \varphi$$

$$\frac{a-b}{b} = \varphi - 1$$

$$\frac{b}{a-b} = \frac{1}{\varphi - 1} \cdot \frac{\varphi + 1}{\varphi + 1} = \frac{\varphi + 1}{\varphi^2 - 1} = \frac{\varphi^2}{\varphi} = \varphi$$

$$\Rightarrow \boxed{\frac{x_3 - x_1}{x_2 - x_1} = \frac{x_4 - x_2}{x_4 - x_3} = \varphi}$$

krok					
1	x_i	4	4,763932	5,236068	6
	$f(x_i)$	−41,320	−117,050	−162,74436	−112,724
2	x_i	4,763932	5,236068	5,527864	6
	$f(x_i)$	−117,050	−162,74436	−172,48155	−112,724
3	x_i	5,236068	5,527864	5,7082039	6
	$f(x_i)$	−162,74436	−172,48155	−163,8687	−112,724



c) Z predošlého kroku máme interval $[a,b]$ a bod $c \in [a,b]$ (v prvom kroku volíme $c = \frac{a+b}{2}$ stred intervalu). Konštruujeme kvadratický interpolačný polynóm v bodoch a, c, b a nájdeme jeho minimum ⇝ bod d. Pokračujeme intervalom $[a,c]$ a bodom d, ak $d \in [a,c]$, alebo intervalom $[c,b]$ a bodom d, ak $d \in [c,b]$.

$$\left(\begin{aligned} P(x) &= f[a] + f[a,c](x-a) + f[a,c,b](x-a)(x-c) \\ P'(x) &= f[a,c] + f[a,c,b](2x-a-c) = 0 \\ &\Rightarrow x = \frac{1}{2}\left(a + c - \frac{f[a,c]}{f[a,c,b]}\right) \end{aligned} \right)$$

1. krok: $a=4, \ c=5, \ b=6$

			$f[a,c,b]$
4	-41,320		
		-100,997	
5	-142,317		65,295
		29,593	
6	-112,724		

$$d = \frac{1}{2}\left(4+5+\frac{100,997}{65,295}\right) = 5,27339$$

2. krok $a=5, \ c=5,27339, \ b=6$

			$f[a,c,b]$
5	-142,317		
		-83,611	
5,27339	-165,175		155,7969
		72,186	
6	-112,724		

$$d = \frac{1}{2}\left(10,27339 + \frac{83,611}{155,7969}\right) = 5,4050283$$

3. krok $a=5,273, \ c=5,405, \ b=6$

			$f[a,c,b]$
5,273	-165,175		
		-45,986	
5,405	-171,245		198,54281
		98,355	
6	-112,724		

$$d = 5,4548088$$

d) ak f má dostatok derivácií a poznáme ich, potom môžeme hľadať stacionárny bod f, teda hľadáme koreň rovnice $f'(x)=0$ ak použijeme Newtonovu metódu: $x_{k+1} = x_k - \dfrac{f'(x_k)}{f''(x_k)}$

$$f(x) = e^x \sin x$$
$$f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$
$$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x$$

$$g(x) = x - \frac{f'(x)}{f''(x)} = x - \frac{\cancel{e^x}(\sin x + \cos x)}{2\cancel{e^x}\cos x} = x - \frac{1}{2}(\tan x + 1)$$

$$x_0 = \frac{a+b}{2} = \frac{4+6}{2} = 5$$

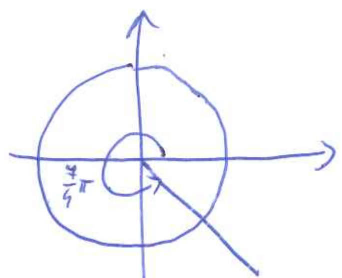
$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 - \frac{1}{2}(\tan x_0 + 1) = 6{,}1902575$$

$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = x_1 - \frac{1}{2}(\tan x_1 + 1) = 5{,}7368556$$

$$x_3 = x_2 - \frac{f'(x_2)}{f''(x_2)} = x_2 - \frac{1}{2}(\tan x_2 + 1) = 5{,}5408889$$

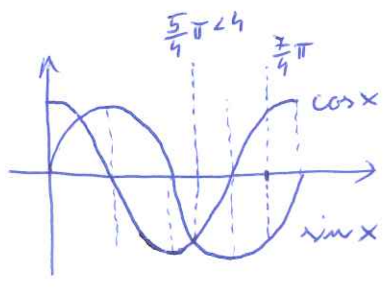
Presná hodnota minima:

$$f'(x) = e^x(\sin x + \cos x) = 0 \iff \sin x + \cos x = 0$$
$$\sin x = -\cos x, \quad x \in [4, 6)$$



$$x = \frac{7}{4}\pi \approx 5{,}4977871$$

	$[4, \frac{7}{4}\pi]$	$[\frac{7}{4}\pi, 6]$
f'	−	+
f	↘	↗



$\frac{7}{4}\pi$ je globálne minimum