rád konvergencie postupnosti $\{x_n\}_{n=0}^{\infty}$ je reálne číslo $p \geq 1$ také, že $\lim\limits_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|} = C \neq 0$

ak x_n vznikla nejakou iteračnou metódou, povieme, že táto metóda je rádu p

pre dostatočne veľké n platí $|x_{n+1}-\xi| \approx C\cdot|x_n-\xi|^p$

vyšetríme rád konvergencie iteračnej metódy

$g(x) = x - Mf(x) \qquad x_{n+1} = x_n - Mf(x_n)$

$\varepsilon_{n+1} = x_{n+1}-\xi \qquad \varepsilon_n = x_n-\xi$

podľa Taylorovej vety platí:

$f(x_n) = f(\xi) + f'(a(x_n))(x_n-\xi)$ $\qquad a(x_n)$ leží medzi x_n a ξ

$f(\xi)=0$ (ξ je koreň f)

$$\begin{aligned}\varepsilon_{n+1} = x_{n+1}-\xi &= x_n - Mf(x_n) - \xi \\ &= x_n-\xi - M f'(a(x_n))(x_n-\xi) \\ &= (x_n-\xi)(1-Mf'(a(x_n))) \\ &= \varepsilon_n(1-Mf'(a(x_n)))\end{aligned}$$

$$\lim_{n\to\infty}\frac{|\varepsilon_{n+1}|}{|\varepsilon_n|} = \lim_{n\to\infty}\frac{|\varepsilon_n|\cdot|1-Mf'(a(x_n))|}{|\varepsilon_n|} =$$

$$= \lim_{n\to\infty}|1-Mf'(a(x_n))| = 1-Mf'(\xi) = g'(\xi) > 0$$

$(x_n\to\xi \Rightarrow a(x_n)\to\xi)$

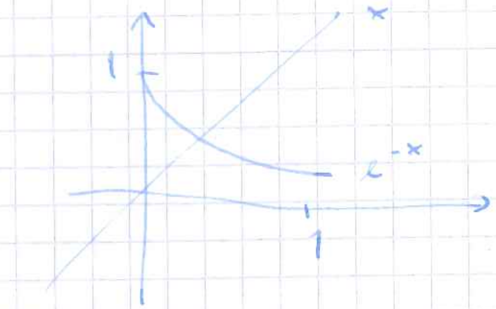
rád konvergencie je evidentne $p=1$

$\Rightarrow$ metóda konverguje lineárne

metóda $x_{k+1} = g(x_k)$ je rádu p práve vtedy, keď stupeň prvej nenulovej derivácie funkcie g v bode ξ je p.

$(\xi = g(\xi),\ g^{(j)}(\xi) = 0$ pre $1 \leq j < p,\ g^{(p)}(\xi) \neq 0)$

príklad: prostou iteračnou metódou riešte rovnicu $f(x) = x - e^{-x} = 0$. Vyšetrite rád konvergencie zvolených iteračných funkcií.



$f(x) = 0 \Leftrightarrow x = e^{-x} = g(x)$

~~$\xi\in[0,1]$~~ $\quad \xi \in [0,1]$

g je vhodná iteračná funkcia napr. na $[\frac{1}{10}, 1]$.

(metóda konverguje pre akúkoľvek počiatočnú podmienku $x_0 \in [0,1]$)

$g'(x) = -e^{-x} \neq 0$ na celom $\mathbb{R}$ $\Rightarrow g'(\xi) \neq 0$

$\Rightarrow g$ je prvého rádu

g môžeme použiť, ale konvergencia je pomalá

$x = e^{-x} \quad / \cdot e^x$

$xe^x = 1 \quad / +x$

$xe^x + x = x+1$

$x(e^x+1) = x+1$

$x = \frac{x+1}{e^x+1} = g_0(x)$

$g_0(0) = \frac{1}{2} \in [0,1]$

$g_0(1) = \frac{2}{e^1+1} \in [0,1]$

$g_0'(x) = \frac{e^x+1-e^x(x+1)}{(e^x+1)^2} = \frac{1-e^x x}{(e^x+1)^2}$

$g_0'(\xi) = 0$

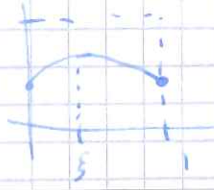
$g_0'(x) > 0$ pre $x\in[0,\xi)$

$g_0'(x) < 0$ pre $x\in(\xi,1]$

g_0 má v ξ lokálny extrém (maximum)

$g_0(\xi) = \frac{\xi+1}{e^\xi+1} = \frac{e^\xi\xi + e^\xi}{e^\xi(e^\xi+1)} = \frac{1+e^\xi}{e^\xi(1+e^\xi)} = \frac{1}{e^\xi} \in [0,1]$

$g_0([0,1]) \subseteq [0,1]$



$g_0([0,\xi]) \subseteq [0,1]$

$g_0([\xi,1]) \subseteq [0,1]$

$\Rightarrow g_0([0,1]) \subseteq [0,1]$

g_0 je kontrakcia na $[0,1]$:

$$g_0''(x) = \frac{(e^x+1)^2(-e^x x - e^x) - 2(1-xe^x)e^x(e^x+1)}{(e^x+1)^4}$$

$$= \frac{(e^x+1)(-e^x x - e^x) - 2e^x(1-xe^x)}{(e^x+1)^3}$$

$$= \frac{-\cancel{e^{2x}x} - e^{2x} - e^x x - \cancel{e^x} - 2e^x + 2xe^{2x}}{(e^x+1)^3}$$

$$= \frac{xe^{2x} - e^{2x} - e^x x - 3e^x}{(e^x+1)^3}$$

$> 0 \quad \forall x \in \mathbb{R}$

ukážeme, že $g_0'' < 0$ na $[0,1]$

$$xe^{2x} - xe^x - e^{2x} - 3e^x < 0$$

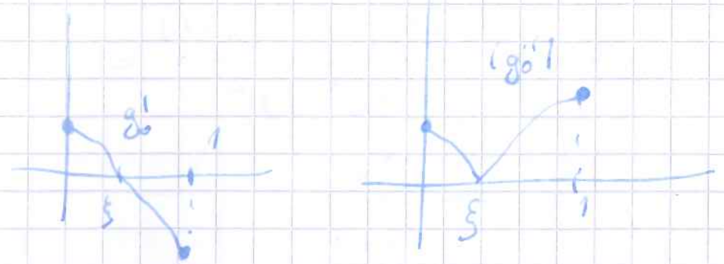
$$xe^x(e^x-1) - e^x(e^x+3) < 0$$

$$xe^x(e^x-1) < e^x(e^x+3)$$

$$\frac{x(e^x-1)}{e^x+3} < 1 \qquad \begin{matrix} e \geq \\ 4 \leq \end{matrix}$$

ekv. úpravy

$\Rightarrow$ g_0'' je záporná na $[0,1]$ $\Rightarrow$ g_0' klesá na $[0,1]$



$$\max_{x\in[0,1]} |g_0'(x)| = \max\{|g_0'(0)|, |g_0'(1)|\} = \max\left\{\frac{1}{4}, \frac{e-1}{(e+1)^2}\right\} < 1$$

g_0 je vhodná iteračná funkcia na $[0,1]$

a ~~tak~~ keďže $g_0'(\xi) = 0$ je metóda aspoň rádu 2

(aspoň druhého rádu)

príklad. Pomocou Newtonovej metódy približne vypočítajte najmenší kladný koreň funkcie $f(x) = \cos x$.

Newtonova metóda: $g(x) = x - \frac{f(x)}{f'(x)}$

zrejme platí $\quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

$f(x) = 0 \iff g(x) = x$

$\xi = \frac{\pi}{2} \qquad f'(x) = -\sin x \Rightarrow g(x) = x + \frac{\cos x}{\sin x} = x + \operatorname{cotan} x$

veta (2.8): $f \in C^2[a,b],\ \xi \in [a,b],\ f'(\xi) \neq 0$

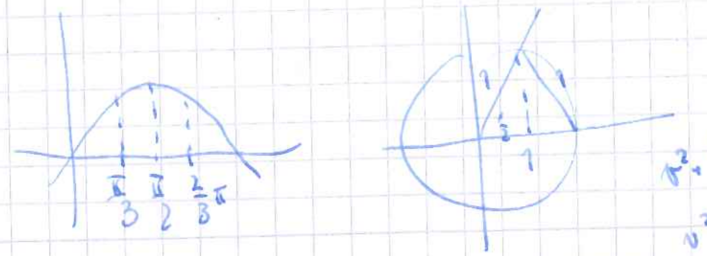
$\Rightarrow \exists \delta > 0,\ \forall x_0 \in [\xi - \delta, \xi + \delta] \subseteq [a,b] \quad \{x_n\}_{n=0}^{\infty}$ konvergujú

$\Rightarrow$ metóda je druhého rádu

vo všeobecnosti nie je jednoduché nájsť toto okolie, ale v tomto prípade sa to dá.

$g'(x) = 1 - \frac{1}{\sin^2 x} \qquad$ zvolíme $I = [\frac{\pi}{3}, \frac{2}{3}\pi]$

$\forall x \in I$ platí: $\quad \frac{\sqrt{3}}{2} \leq \sin x \leq 1 \quad |^2$



$\frac{3}{4} \leq \sin^2 x \leq 1$

$1 \leq \frac{1}{\sin^2 x} \leq \frac{4}{3}$

$v^2 + \frac{1}{4} = 1$

$v^2 = \frac{3}{4} \Rightarrow v = \frac{\sqrt{3}}{2}$

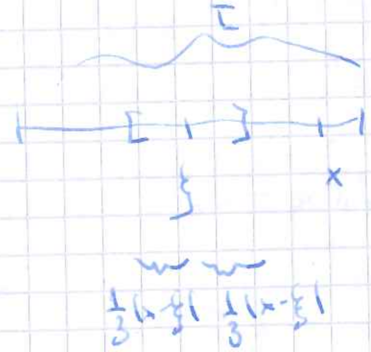
$-\frac{4}{3} \leq -\frac{1}{\sin^2 x} \leq -1$

$-\frac{1}{3} \leq 1 - \frac{1}{\sin^2 x} = g'(x) \leq 0$

$\Rightarrow \forall x \in I$ platí $g'(x) \in [-\frac{1}{3}, 0]$

$\Rightarrow \max_{x \in I} |g'(x)| = \frac{1}{3} < 1 \qquad$ platí $g(I) \subseteq I$?

nech $x \in I \Rightarrow |g(x) - \xi| = |g(x) - g(\xi)| = |g'(\alpha)| \cdot |x - \xi| \leq \frac{1}{3}|x - \xi| \qquad \alpha \in I$



I je symetrický okolo ξ

$\Rightarrow g: I \to I$

g je zrejme spojitá na I

$x_0 = \frac{\pi}{3}$

$x_1 = g(x_0) = \frac{\pi}{3} + \operatorname{cotan} \frac{\pi}{3} = \frac{\pi}{3} + \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\pi}{3} + \frac{1}{\sqrt{3}} \approx 1{,}62454782$

$x_2 = g(x_1) = \frac{\pi}{3} + \frac{1}{\sqrt{3}} + \operatorname{cotan}\left(\frac{\pi}{3} + \frac{1}{\sqrt{3}}\right) \approx 1{,}5707445$

$\cdot 2 \approx 3{,}141489$

ch. $\approx 0{,}0001$

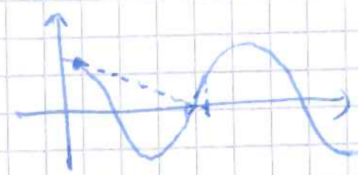
$x_0 = \frac{3}{2}$

$x_1 = g(x_0) = \frac{3}{2} + \operatorname{cotan} \frac{3}{2} \approx 1{,}57091484$

$x_2 = g(x_1) \approx 1{,}57079633$

$\cdot 2 \approx 3{,}14159265 \quad$ chyba je rádovo 10^{-12}

čo sa stane ak nevhodne zvolíme počiatočnú podmienku?



Hľadáme x_0 tak, aby $x_0 - \frac{g(x_0)}{g'(x_0)} = x_0 + 2\pi$

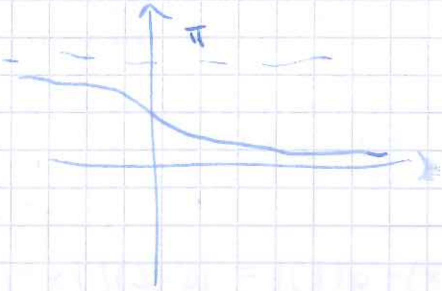
$$x_0 + \operatorname{cotan} x_0 = x_0 + 2\pi$$

$$\operatorname{cotan} x_0 = 2\pi$$

$$x_0 = \operatorname{arccotan}(2\pi)$$

$$x_0 \simeq 0{,}15783119$$

$g'(x) = 1 - \frac{1}{\sin^2 x}$



$$\cos^2(\operatorname{arccotan}(\cancel{2\pi}\,x)) + \sin^2(\operatorname{arccotan}(\cancel{2\pi}\,x)) = 1$$

$$x^2 + 1 = \frac{1}{\sin^2(\operatorname{arccotan}(x))}$$

$$\sin^2(\operatorname{arccotan}(x)) = \frac{1}{x^2+1}$$

$\cancel{g(x_0)}$ $g'(x_0) = 1 - x_0^2 - 1 = -x_0^2 = -4\pi^2$

Hľadáme x_0 tak, aby $x_0 - \frac{g(x_0)}{g'(x_0)} = 2\pi \cancel{x_0} - x_0$

$$2x_0 + \operatorname{cotan} x_0 = 2\pi$$

Fourierove podmienky:

$f \in C^2[a,b]$

f má jediný koreň ξ v $[a,b]$

f', f'' nemenia znamienko na $[a,b]$, $f'(x) \neq 0 \ \forall x \in [a,b]$

$x_0 \in \{a,b\}$ tak, aby $f(x_0)\cdot f'' > 0$

$\Rightarrow$ Newtonova metóda konverguje monotónne

Príklad. Pomocou Newtonovej metódy nájdite približnú hodnotu kladného koreňa funkcie $f(x) = e^{-x^2} - \frac{1}{4}$.

$f'(x) = -2xe^{-x^2} < 0 \quad \forall x \in (0,\infty)$

$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 2e^{-x^2}(\cancel{4}2x^2 - 1)$

$2x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{2}}$ $\quad \pm\frac{1}{\sqrt{2}}$ sú inflexné body

$\Rightarrow f''(x) \geq 0 \quad \forall x \in [\frac{1}{\sqrt{2}}, \infty)$

$f(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{e}} - \frac{1}{4} > 0$ lebo $\sqrt{e} < 4$

$f(2) = \frac{1}{e^4} - \frac{1}{4} < 0$ lebo $4 < e^4$

$\Rightarrow f$ má v $[\frac{1}{\sqrt{2}}, 2]$ jediný koreň

volíme $x_0 = \frac{1}{\sqrt{2}}$ .. $x_1 = \frac{1}{\sqrt{e}} - \frac{1}{4}$, $x_2 = g(x_1)$ ....

doplnok: metóda tretieho rádu

Newtonova metóda: $f(x) \simeq f(x_n) + f'(x_n)(x - x_n)$

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

$$\leadsto x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) \simeq f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2}(x - x_n)^2$$

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)^2$$

$$0 = \frac{f''(x_n)}{2}x_{n+1}^2 + \left(f'(x_n) - f''(x_n)x_n\right)x_{n+1} + \frac{f''(x_n)}{2}x_n^2 - f'(x_n)x_n + f(x_n)$$

$$D = \left(f'(x_n) - f''(x_n)x_n\right)^2 - 4\frac{f''(x_n)}{2}\left(\frac{f''(x_n)}{2}x_n^2 - f'(x_n)x_n + f(x_n)\right)$$

$$= f'(x_n)^2 - 2f(x_n)f''(x_n) > 0$$

$$(x_{n+1})_{1,2} = \frac{f''(x_n)x_n - f'(x_n) \pm \sqrt{f'(x_n)^2 - 2f(x_n)f''(x_n)}}{f''(x_n)}$$

$$(x_{n+1})_{1,2} = x_n + \frac{-f'(x_n) \pm \sqrt{f'(x_n)^2 - 2f(x_n)f''(x_n)}}{f''(x_n)}$$

$$g(x) = x + \frac{-f'(x) \pm \sqrt{f'^2(x) - 2f(x)f''(x)}}{f''(x)}$$

potrebujeme $f(x)=0 \Leftrightarrow g(x)=x \rightarrow$

$f'(\xi)>0$	$f'(\xi)<0$
$+$	$-$

$f \in C^4[a,b]$, $f'(\xi)\neq 0$, $f''(\xi)\neq 0$, $\xi\in[a,b]$

nech $f'(\xi)>0$:

$$g(x) = x + \frac{-f'(x) + \sqrt{f'^2(x) - 2f(x)\cdot f''(x)}}{f''(x)}$$

$$\cancel{g'(x) = 1 + \frac{-f'(x) + \frac{1}{2}\frac{1}{\sqrt{f'^2(x)-2f(x)f''(x)}}\cdot(2f}{}}$$

$$g' = 1 + \frac{\left(-f'' + \frac{1}{2}\frac{1}{\sqrt{f'^2-2ff''}}\cdot(2f'f''-2f'f''-2ff''')\right)f'' - \left(-f'+\sqrt{f'^2-2ff''}\right)f'''}{f''^2}$$

$$g' = 1 + \frac{\left(-f'' - \frac{ff'''}{\sqrt{f'^2-2ff''}}\right)f'' - \left(-f'+\sqrt{f'^2-2ff''}\right)f'''}{f''^2}$$

$$g' = 1 + \frac{f'f''' - f''f''}{f''^2} - \frac{ff''f''' + f'^2f''' - 2ff''f'''}{\sqrt{f'^2-2ff''}\cdot f''^2} =$$

$$= 1 + \frac{f'f''' - f''^2}{f''^2} - \frac{f'^2f''' - ff''f'''}{\sqrt{f'^2-2ff''}\cdot f''^2}$$

$$g'(\xi) = \cancel{1 + \frac{f'(\xi)f'''(\xi)}{f''^2(\xi)} - 1 - \frac{f'^2(\xi)f'''(\xi)}{\sqrt{f'^2(\xi)}\cdot f''^2(\xi)}}$$

$$= \cancel{1 - \frac{f'(\xi)}{f''(\xi)} - \frac{f'(\xi)f'''(\xi)}{f''^2(\xi)}}$$

$g'(\xi) = 0$

$$\cancel{g'(\xi)}\quad g' = \frac{f'f'''}{f''^2} - \frac{f'^2f''' - ff''f'''}{\sqrt{f'^2-2ff''}\cdot f''^2}$$

$$g'' = \frac{(f''f''' + f'f^{IV})f''^2 - 2f'f'''^2f''}{f''^4}$$

$$- \frac{(2f'f''f''' + f'^2f^{IV} - f'f''f''' - ff'''^2 - ff''f^{IV})\sqrt{f'^2-2ff''}\cdot f''^2 - (f'^2f''' - ff''f''')\left(\frac{1}{2}\frac{-ff''^2f'''}{\sqrt{f'^2-2ff''}} + 2f''f'''\sqrt{f'^2-2ff''}\right)}{(f'^2-2ff'')f''^4}$$

$$g''(\xi) = \frac{(f''f''' + f'f^{IV})f''^2 - 2f'f'''^2f''}{f''^4} - \frac{(f'f''f''' + f'^2f^{IV})f'f''^2}{f'^2f''^4}$$

$$- \frac{f'^2f'''\cdot 2f''f'''f'}{f'^2f''^3}$$

$$= \frac{f''f''' + f'f^{IV}}{f''^2} - \frac{2f'f'''^2}{f''^3} - \frac{f''f''' + f'f^{IV}}{f''^2} - \frac{2f'f'''^2}{f''^3}$$

$g''(\xi) = 0$

pomocou tejto metódy určíme približne hodnotu e

$f(x) = \ln x - 1$

$f'(x) = \frac{1}{x} > 0 \quad \forall x\in(0,\infty)$

$f''(x) = -\frac{1}{x^2} < 0 \quad \forall x\in(0,\infty)$

keďže $e<3$

$\sqrt{e}<\sqrt{3}$

volíme $x_0 \geq \sqrt{3}$

$f'^2(x) - 2f(x)f''(x) > 0$

$$\frac{1}{x^2} + \frac{2(\ln x - 1)}{x^2} =$$

$$= \frac{2\ln x - 1}{x^2} > 0$$

$\Leftrightarrow 2\ln x - 1 > 0$

$\ln x > \frac{1}{2}$

$\Rightarrow x > \sqrt{e}$

Pomocou metódy sečníc ~~a quasi-Newtonovej metódy~~ nájdite záporný koreň funkcie $f(x)=2x+3\cos x-e^x$

metóda sečníc:

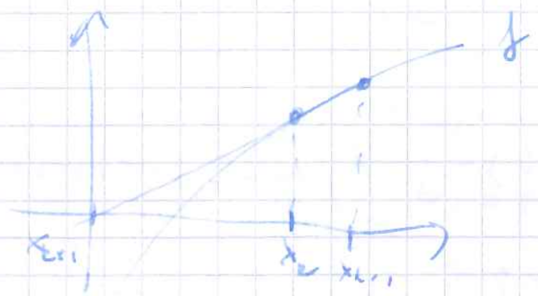
deriváciu $f'(x_k)$ aproximujeme výrazom $\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$

(potrebujeme dve počiatočné hodnoty x_0, x_1)

$$x_{k+1} = x_k - \frac{x_k-x_{k-1}}{f(x_k)-f(x_{k-1})} f(x_k)$$ → metóda je dvojkroková

geometrická interpretácia:

x_{k+1} je priesečník sečnice prechádzajúcej bodmi $[x_k, f(x_k)]$ a $[x_{k-1}, f(x_{k-1})]$ s osou x



ak f' a f'' sú spojité v okolí ξ, $f'(\xi)\neq 0$

⇒ metóda konverguje pre x_0, x_1 blízko ξ

rád je $\frac{1+\sqrt{5}}{2}$

$f(x)=2x+3\cos x-e^x$ má derivácie všetkých rádov

$f(0) = 3-1 = 2 > 0$

$f(-3) = -6 + \underbrace{3\cos(-3)}_{\leq 3} \underbrace{-e^{-3}}_{<0} < 0$

f má v intervale $[-3,0]$ koreň

$f'(x) = 2-3\sin x-e^x = \underbrace{2-e^x}_{>0 \text{ pre } x\leq 0} \underbrace{-3\sin x}_{\geq 0 \text{ pre } x\in[-\pi,0]} > 0$ pre $x\in[-3,0]$

⇒ $f'(\xi)>0$ ⇒ $f'(\xi)\neq 0$

$x_0=-3,\ x_1=-2$

$$x_2 = x_1 - \frac{x_1-x_0}{f(x_1)-f(x_0)} f(x_1) = -2 - \frac{f(-2)}{f(-2)-f(-3)} =$$

$$= -2 - \frac{-4+3\cos(-2)-e^{-2}}{2+3\cos(-2)-3\cos(-3)-e^{-2}+e^{-3}}$$

$$\approx -0{,}519309\ldots$$

$$x_3 = x_2 - \frac{x_2-x_1}{f(x_2)-f(x_1)} \cdot f(x_2) \approx -0{,}745544\ldots$$

príklad 2. Pomocou ~~quasi Newtonovej metódy a~~ metódy regula falsi nájdite kladný koreň funkcie $f(x) = 2x\cos(2x) - (x-2)^2$.

metóda regula falsi:

$$x_{k+1} = x_k - \frac{x_k - x_s}{f(x_k) - f(x_s)} \cdot f(x_k)$$

$s = s(k) < k$ je najväčší index taký, že $f(x_k) \cdot f(x_s) < 0$.

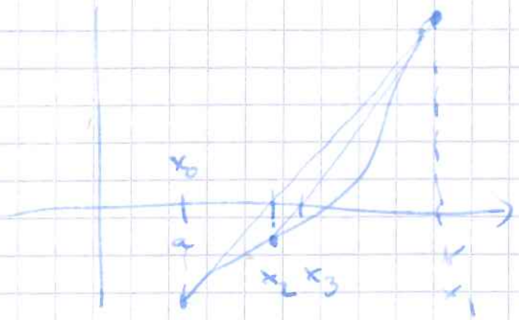
$f \in C[a,b]$, $f(a) \cdot f(b) < 0$, ξ je jediný koreň v $[a,b]$

$\Rightarrow$ metóda konverguje pre každé $x_0, x_1 \in [a,b]$, $f(x_0) \cdot f(x_1) < 0$.

geometrická interpretácia:

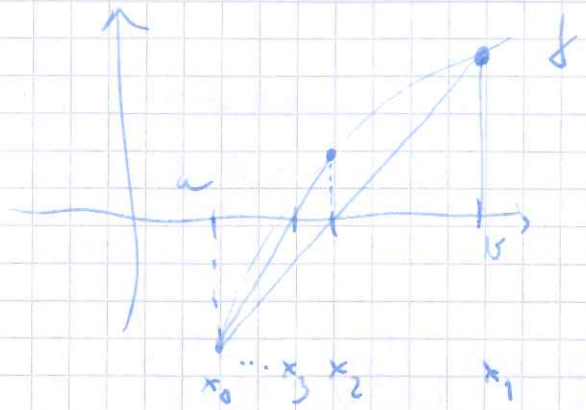
$k \geq 1$ $\quad a_1 = a, b_1 = b$

na intervale ~~aproximujeme~~ $[a_k, b_k]$ aproximujeme funkciu f priamkou prechádzajúcou bodmi $[a_k, f(a_k)]$, $[b_k, f(b_k)]$. ~~Bod~~ x_{k+1} je prienik tejto priamky s osou x. Interval $[a_{k+1}, b_{k+1}]$ je ten z intervalov $[a_k, x_{k+1}]$, $[x_{k+1}, b_k]$, v ktorého krajných bodoch má f opačné znamienko.



ak f'' nemení znamienko na $[a,b]$ $\Rightarrow$ jeden z bodov x_0, x_1 je pevný (všetky sečnice vychádzajú z neho). Je to ten, v ktorom je znamienko f totožné so znamienkom f'')

$\Rightarrow$ metóda v takom prípade konverguje monotónne



$f(x) = 2x\cos(2x) - (x-2)^2$ má všetky derivácie na $\mathbb{R}$ $(f \in C^\infty(\mathbb{R}))$

$f\left(\frac{\pi}{2}\right) = -\pi - \left(\frac{\pi}{2} - 2\right)^2 < 0$

$f(\pi) = 2\pi - (\pi - 2)^2 > 0$ $\Rightarrow$ f má v $\left[\frac{\pi}{2}, \pi\right]$ koreň

$x_0 = \frac{\pi}{2}$ $\quad x_1 = \pi$

$s(1) = 0$ $\quad f(x_0) < 0$

$s(2) = 1$ $\quad f(x_1) > 0$

$s(3) = 2$ $\quad f(x_2) < 0$

$f(x_3) > 0$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) =$$

$$= \pi - \frac{\frac{\pi}{2}}{2\pi - (\pi-2)^2 + \pi + \left(\frac{\pi}{2} - 2\right)^2} \left(2\pi - (\pi-2)^2\right)$$

$\doteq 2{,}1997776$ $\qquad f(x_2) \doteq -1{,}393801$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) \doteq 2{,}4057436$$

$f(x_3) \doteq 0{,}3114$

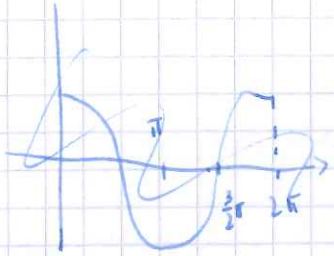
$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) \ \ldots$$

$f'(x) = 2\cos 2x - 4x\sin 2x - 2(x-2)$

~~aby sme mohli použiť práve Newtonovu metódu potrebujeme $f'(\xi) \neq 0$~~. Ukážeme, že $f'(x) > 0$ na intervale, v ktorom leží koreň.

To implikuje, že f je monotónna, a teda má v danom intervale len jeden koreň.

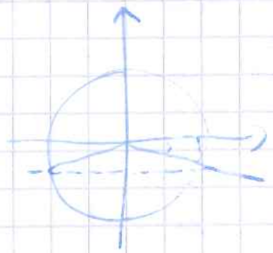
v intervale $I = [2, \frac{11}{12}\pi]$ leží koreň f:



$f(2) = 4\cos 4 < 0$

$f(\frac{11}{12}\pi) = \frac{11}{6}\pi \cdot \cos(\frac{11}{6}\pi) - (\frac{11}{12}\pi - 2)^2 > 0$

$\Rightarrow \xi \in [2, \frac{11}{12}\pi]$



$x \in [2, \frac{11}{12}\pi] \Rightarrow 2x \in [4, \frac{11}{6}\pi]$
$\subseteq [\frac{7}{6}\pi, \frac{11}{6}\pi]$

ak $2x \in [4, \frac{11}{6}\pi] \Rightarrow \sin 2x \in [-1, -\frac{1}{2}]$

$\sin 2x \leq -\frac{1}{2} \quad / \cdot 2x$

$2x\sin 2x \leq -x \quad / \cdot (-2)$

$-4x\sin 2x \geq 2x \geq 4$

$\Rightarrow -2(x-2) = -2x+4 \in [4 - \frac{11}{6}\pi, 0]$

$36:11 = 3,2$

$-2x+4 > -2 \qquad 4 - \frac{11}{6}\pi > -2$

$\frac{11}{6}\pi < 6$

$11\pi < 36$

$\pi < \frac{36}{11}$

$2\cos 2x \in [-2, 2]$

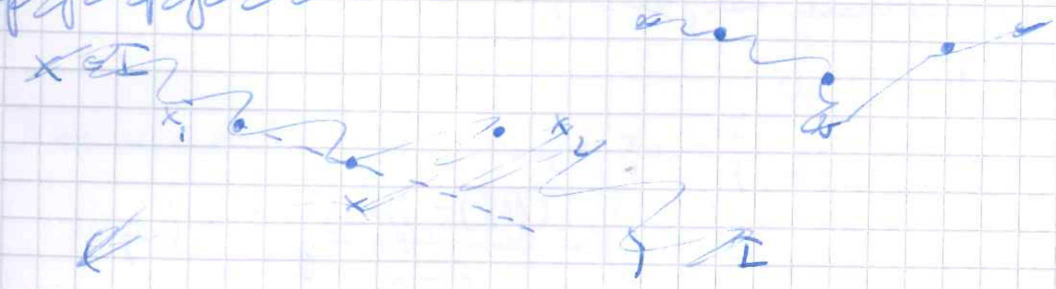
$2\cos 2x \geq -2$

$f'(x) > 0$ pre $x \in [2, \frac{11}{12}\pi] \ni \xi$

$\Rightarrow f'(\xi) \neq 0$

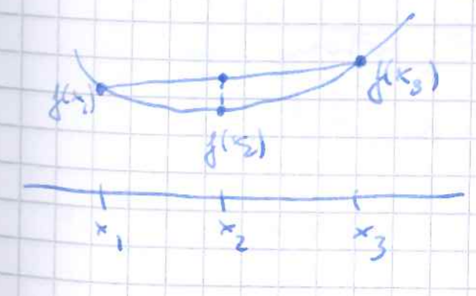
Príklad. Nech f je konvexná funkcia na intervale $I=(a,b)$ majúca v I jediný koreň. Ukážte, že existujú $x_0, x_1 \in I$ také, že metóda sečníc generuje monotónnu postupnosť konvergujúcu k tomuto koreňu.

~~f je spojitá na I:~~



konvexnosť f na I: $\forall x_1, x_2, x_3 \in I$ také, že $x_1 < x_2 < x_3$ platí

$$f(x_2) \leq f(x_1) + \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x_2 - x_1)$$



$\exists\, x_0 \in [a,b]$ tak, že $f(x_0) > 0$

dokážeme sporom: $\forall\, x \in [a,b]$, $f(x) \leq 0$

I. $\forall\, x \in [a,b]$ $f(x) = 0$ triviálne

II. $\exists\, \bar{x} \in [a,b]$ $f(\bar{x}) < 0$

pre určitosť povedzme $\bar{x} < \xi$

zvolme $\tilde{x} \in (\xi, b)$

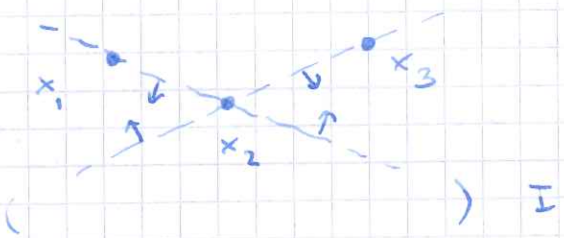
podľa predpokladu $f(\tilde{x}) \leq 0$

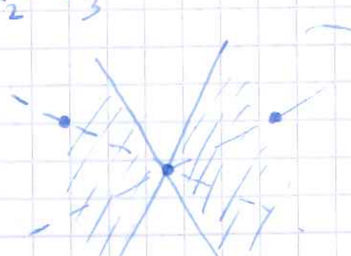
buď $\lambda \in (0,1)$ také, že $\lambda \bar{x} + (1-\lambda)\tilde{x} = \xi$

$$0 = f(\xi) = f(\lambda \bar{x} + (1-\lambda)\tilde{x}) \leq \lambda \underset{<0}{f(\bar{x})} + (1-\lambda)\underset{\leq 0}{f(\tilde{x})} < 0 \quad \text{spor}$$

1) f je spojitá na I: ~~$x_2 \in I$~~ $x_2 \in I$ ľub.

zvoľme $x_1, x_3 \in I$ tak, že $x_1 < x_2 < x_3$





smernice priamok sú $L, -L$

$\Rightarrow$ pre x dostatočne blízke x_2 platí $|f(x) - f(x_2)| \le L|x - x_2|$

$\Rightarrow$ f je spojitá v x_2 (ak $\varepsilon > 0$ potom zvoľme $\delta = \frac{\varepsilon}{L}$ a $\forall x, |x - x_2| < \delta$ platí $|f(x) - f(x_2)| \le L|x - x_2| \le L\frac{\varepsilon}{L} = \varepsilon$)

$\Rightarrow$ f nemení znamienko na intervaloch (a, ξ) a (ξ, b)
(je stále kladná alebo stále záporná)
(ak $x_1, x_2 \in (a, \xi)$ a $f(x_1) \cdot f(x_2) < 0$ potom v intervale (x_1, x_2) leží koreň, čo je spor s tým, že ξ je jediný v (a, b).
pre určitosť povedzme, že $f > 0$ na (a, ξ)

2) zvoľme ľubovoľné $x_0, x_1 \in (a, \xi)$ tak, že ~~$f(x_0) > f$~~ $x_0 < x_1 < \xi$
(to, že $x_0, x_1 < \xi$ využijeme tak, že $f(x_0), f(x_1) > 0$)

ukážeme, že $f(x_0) > f(x_1)$

z konvexnosti: $f(x_1) \le f(x_0) + \frac{f(\xi) - f(x_0)}{\xi - x_0}(x_1 - x_0)$

$$= f(x_0) - \frac{\overbrace{f(x_0)}^{>0}}{\underbrace{\xi - x_0}_{>0}} \underbrace{(x_1 - x_0)}_{>0} < f(x_0)$$

ukážeme, že ak $x_0 < x_1 < x_2 < \dots < x_k < \xi$, potom je x_{k+1} definované a platí $x_0 < \dots < x_k < x_{k+1} < \xi$.

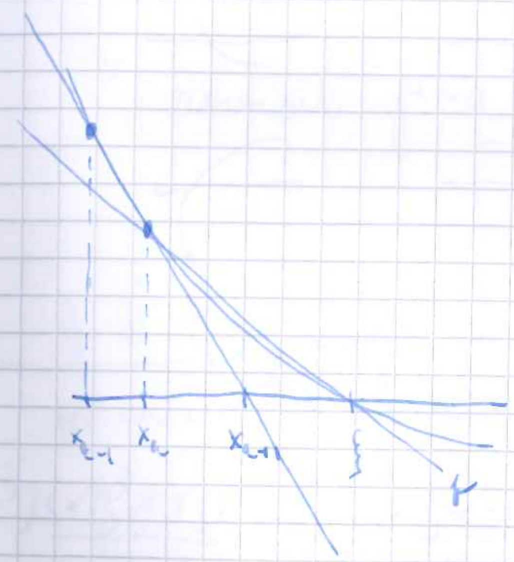
zrejme platí $f(x_0) > f(x_1) > \dots > f(x_k) > 0$ (rovnaký argument ako pre x_0 a x_1)

$x_i < x_{i+1} < \xi \qquad i \in \{0, \dots, k-1\}$

$$f(x_{i+1}) \le f(x_i) + \frac{f(\xi) - f(x_i)}{\xi - x_i}(x_{i+1} - x_i)$$

$$= f(x_i) - \frac{\overbrace{f(x_i)}^{>0}}{\underbrace{\xi - x_i}_{>0}} \underbrace{(x_{i+1} - x_i)}_{>0} < f(x_i)$$

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k) \Leftrightarrow x_{k+1} - x_k = -\frac{\overbrace{x_k - x_{k-1}}^{>0}}{\underbrace{f(x_k) - f(x_{k-1})}_{<0}} \underbrace{f(x_k)}_{>0} > 0$$



$\Rightarrow x_{k+1} > x_k$

rovnica priamky p:

$$y = \frac{f(\xi) - f(x_k)}{\xi - x_k}(x - x_k) + f(x_k)$$

$(\xi, 0)$ leží na p, preto

$$(\xi - x_k)\frac{f(\xi) - f(x_k)}{\xi - x_k} + f(x_k) = 0$$

$$\xi = x_k - \frac{\xi - x_k}{f(\xi) - f(x_k)} f(x_k)$$

z konvexnosti pre $x_{k-1} < x_k < \xi$ platí

$$f(x_k) \le f(x_{k-1}) + \frac{f(\xi) - f(x_{k-1})}{\xi - x_{k-1}}(x_k - x_{k-1})$$

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \le \frac{f(\xi) - f(x_{k-1})}{\xi - x_{k-1}}$$

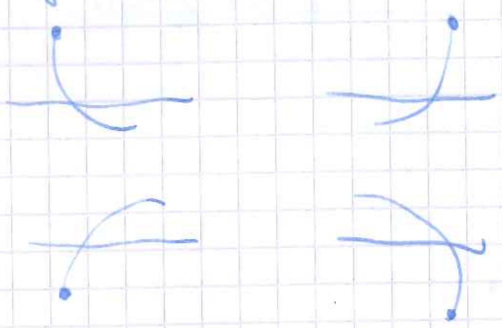
$$\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \ge \frac{\xi - x_{k-1}}{f(\xi) - f(x_{k-1})}$$

$$-\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \le -\frac{\xi - x_{k-1}}{f(\xi) - f(x_{k-1})}$$

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})} f(x_k) \le x_k - \frac{\xi - x_{k-1}}{f(\xi)-f(x_{k-1})} f(x_k) = \xi$$

ukázali sme, že $x_0 < x_1 < \dots < x_k < x_{k+1} \le \xi$

príklad: nech $f \in C[a,b]$, $f(a)\cdot f(b) < 0$ a nech f'' nemení na $[a,b]$ znamienko. Vieme, že za týchto predpokladov je metóda regula falsi v skutočnosti jednokroková. Vypočítajte rád konvergencie.



$x_0 \in \{a,b\}$ taký, že $f(x_0)\cdot f'' > 0$

$$\Rightarrow x_{k+1} = x_k - \frac{x_k - x_0}{f(x_k)-f(x_0)} f(x_k)$$

Iteračná funkcia má tvar $g(x) = x - \frac{x - x_0}{f(x)-f(x_0)} f(x)$

$\xi \in [a,b]$ koreň

$f(\xi) = 0$

$$= \frac{x f(x) - x f(x_0) - x f(x) + x_0 f(x)}{f(x)-f(x_0)}$$

$$= \frac{x_0 f(x) - x f(x_0)}{f(x)-f(x_0)}$$

$$g'(x) = \frac{(x_0 f'(x) - f(x_0))(f(x)-f(x_0)) - f'(x)(x_0 f(x) - x f(x_0))}{(f(x)-f(x_0))^2}$$

$$= \frac{x_0 f'(x) f(x) - x_0 f(x_0) f'(x) - f(x_0) f(x) + f^2(x_0) - x_0 f'(x) f(x) + f(x_0) x f'(x)}{(f(x)-f(x_0))^2}$$

$=$

$$g'(x) = 1 - \frac{f(x)-f(x_0) - f'(x)(x-x_0)}{(f(x)-f(x_0))^2} f(x) - \frac{x - x_0}{f(x)-f(x_0)} f'(x)$$

$$g'(\xi) = 1 + \frac{\xi - x_0}{f(x_0)} f'(\xi)$$

povedzme, že $f'' \ge 0$, $f(x_0) > 0$ $x_0 = a$

~~f'' rastúca na $[a,b]$~~ $\Rightarrow f' \in C[a,b]$

podľa Lagrangeovej vety existuje $c \in (x_0, \xi)$ taký, že

$$f'(c) = \frac{f(\xi) - f(x_0)}{\xi - x_0} \qquad c < \xi \Rightarrow f'(c) \le f'(\xi)$$

$$f'(\xi) > \frac{f(\xi) - f(x_0)}{\xi - x_0} \Rightarrow f'(\xi)(\xi - x_0) > f(\xi) - f(x_0) = -f(x_0)$$

$$\frac{f'(\xi)(\xi - x_0)}{f(x_0)} > -1$$

rovnosť $f'' = 0$ neplatí na žiadnom podintervale

$$g'(\xi) = f'(\xi)\frac{\xi - x_0}{f(x_0)} + 1 > 0$$

g je prvého rádu

pokračovanie predošlého príkladu: a) pre každé k, $x_k < \xi$:

3) ukážeme, že $x_k \to \xi$

$\{x_k\}_{k=0}^{\infty}$ je rastúca postupnosť zhora ohraničená ξ

$\Rightarrow$ existuje limita $\mu \le \xi$

$x_k \to \xi$ dokážeme sporom: nech $\mu < \xi$ $\Rightarrow f(\mu) > 0$

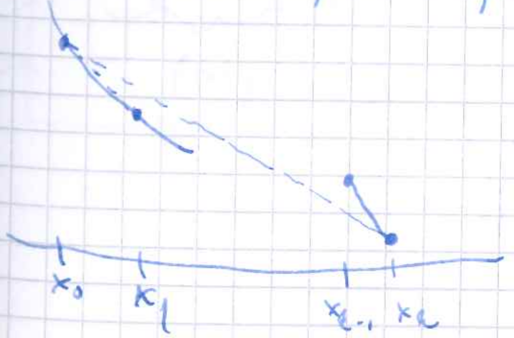
$$\underbrace{x_{k+1} - x_k}_{\to 0 \text{ pre } k \to \infty} = -\frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})} \underbrace{f(x_k)}_{\to f(\mu) \neq 0} \quad (f \text{ je spojitá})$$

$\Rightarrow$

$|x_{k+1} - x_k| \le |x_{k+1} - \mu| + |x_k - \mu| \to 0$

$$\Rightarrow \underbrace{\frac{x_k - x_{k-1}}{f(x_k)-f(x_{k-1})}}_{>0} \to 0 \text{ pre } k \to \infty$$

$$\frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}} \to -\infty$$



$\forall k \geq 2 \qquad x_0 < x_1 < x_k$

$$\underbrace{\frac{f(x_1)-f(x_0)}{x_1-x_0}}_{<0} \leq \frac{f(x_k)-f(x_0)}{x_k-x_0}$$

$\Rightarrow \exists k \in \mathbb{N},\ k \geq 2 \qquad \frac{x_1-x_0}{f(x_1)-f(x_0)} < \frac{x_k-x_{k-1}}{f(x_k)-f(x_{k-1})}$

$$\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}} < \frac{f(x_1)-f(x_0)}{x_1-x_0} \leq \frac{f(x_k)-f(x_0)}{x_k-x_0}$$

pre konvexnú funkciu platí $\qquad x_0 < x_{k-1} < x_k$

$$\frac{f(x_k)-f(x_0)}{x_k-x_0} \leq \frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$$, čo je spor