x_i, $i \in \{0, \dots, n\}$ uzly $\quad x_i \neq x_j$ pre $i \neq j$

$f_i = f(x_i)$ funkčné hodnoty

hľadáme polynóm P stupňa nanajvýš n taký, že

$$P(x_i) = f_i \quad \forall i \in \{0, \dots, n\}$$

Lagrangeov tvar interpolačného polynómu:

$$\pi_i(x) = \prod_{j \neq i} (x - x_j), \quad \omega_{n+1}(x) = \prod_{i=0}^{n} (x - x_i)$$

fundamentálne polynómy L_i: $\quad L_i(x_j) = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}$

$$L_i(x) = \frac{\pi_i(x)}{\pi_i(x_i)} = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} = \frac{(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

$$\boxed{P(x) = \sum_{j=0}^{n} f_i L_i(x)}$$

Newtonov tvar interpolačného polynómu:

~~f[x_n]~~ pomerné diferencie:

$$f[x_k] = f_k, \quad f[x_k, x_m] = \frac{f_k - f_m}{x_k - x_m}$$

$$f[x_m, \dots, x_{m+k}] = \frac{f[x_{m+1}, \dots, x_{m+k}] - f[x_m, \dots, x_{m+k-1}]}{x_{m+k} - x_m}$$

$$\boxed{\begin{aligned} P(x) = {} & f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots \\ & \dots + f[x_0, \dots, x_n](x - x_0) \cdots (x - x_{n-1}) \end{aligned}}$$

chyba interpolácie: $f \in C^{n+1}[a, b]$, $x_i \in [a, b]$

$$\Rightarrow \forall \bar{x} \in [a, b] \ \exists \xi = \xi(\bar{x}) \in (a, b): f(\bar{x}) - P(\bar{x}) = \frac{\omega_{n+1}(\bar{x})}{(n+1)!} f^{(n+1)}(\xi)$$

príklad 1. Nájdite Lagrangeov interpolačný polynóm, ak je dané

x_i	0	1	2	5
f_i	2	3	12	147

1. spôsob:

$$L_0(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} = -\frac{1}{10}(x^2-3x+2)(x-5)$$

$$= -\frac{1}{10}(x^3-8x^2+17x-10)$$

$$L_1(x) = \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} = \frac{1}{4}(x^2-2x)(x-5)$$

$$= \frac{1}{4}(x^3-7x^2+10x)$$

$$L_2(x) = \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} = -\frac{1}{6}(x^2-x)(x-5)$$

$$= -\frac{1}{6}(x^3-6x^2+5x)$$

$$L_3(x) = \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} = \frac{1}{60}(x^2-x)(x-2)$$

$$= \frac{1}{60}(x^3-3x^2+2x)$$

$$P(x) = \sum_{i=0}^{3} f_i L_i(x) = 2L_0(x) + 3L_1(x) + 12L_2(x) + 147L_3(x)$$

$$= \left(-\frac{1}{5}+\frac{3}{4}-2+\frac{147}{60}\right)x^3 + \left(\frac{8}{5}-\frac{21}{4}+12-\frac{147}{20}\right)x^2 +$$

$$+\left(-\frac{34}{10}+\frac{30}{4}-10+\frac{147}{30}\right)x + 2$$

$$= x^3+x^2-x+2$$

príklad 1. pokračovanie

2. spôsob:

$$\omega_4(x) = \prod_{j=0}^{n} (x - x_j) = (x-0)(x-1)(x-2)(x-5)$$
$$= (x^2 - x)(x^2 - 7x + 10)$$
$$= x^4 - 8x^3 + 17x^2 - 10x$$

$\pi_i(x)$ a $\pi_i(x_i)$ dopočítame Hornerovou schémou:

$\pi_0(x)$:

$$\begin{array}{c|ccccc} & 1 & -8 & 17 & -10 & 0 \\ \hline 0 & 1 & -8 & 17 & -10 & 0 \end{array} \Rightarrow \pi_0(x) = x^3 - 8x^2 + 17x - 10$$

$\pi_0(0)$:

$$\begin{array}{c|cccc} & 1 & -8 & 17 & -10 \\ \hline 0 & 1 & -8 & 17 & \boxed{-10} \end{array} \qquad \pi_0(0) = -10$$

$\pi_1(x)$:

$$\begin{array}{c|ccccc} & 1 & -8 & 17 & -10 & 0 \\ \hline 1 & 1 & -7 & 10 & 0 & 0 \\ 1 & 1 & -6 & 4 & \boxed{4} & \end{array} \begin{array}{l} \\ \Rightarrow \pi_1(x) = x^3 - 7x^2 + 10x \\ \Rightarrow \pi_1(1) = 4 \end{array}$$

$\pi_2(x)$:

$$\begin{array}{c|ccccc} & 1 & -8 & 17 & -10 & 0 \\ \hline 2 & 1 & -6 & 5 & 0 & 0 \\ 2 & 1 & -4 & -3 & \boxed{-6} & \end{array} \begin{array}{l} \\ \Rightarrow \pi_2(x) = x^3 - 6x^2 + 5x \\ \Rightarrow \pi_2(2) = -6 \end{array}$$

$\pi_3(x)$:

$$\begin{array}{c|ccccc} & 1 & -8 & 17 & -10 & 0 \\ \hline 5 & 1 & -3 & 2 & 0 & 0 \\ 5 & 1 & 2 & 12 & \boxed{60} & \end{array} \begin{array}{l} \\ \Rightarrow \pi_3(x) = x^3 - 3x^2 + 2x \\ \Rightarrow \pi_3(5) = 60 \end{array}$$

$$P(x) = \sum_{j=0}^{n} f_j \frac{\pi_j(x)}{\pi_j(x_j)} = \ldots$$

príklad 2. ~~Vsa~~ Nájdite Newtonov interpolačný polynóm, ak je dané

x_i	0	2	3	5
f_i	1	3	2	5

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_0, x_1, x_2, x_3]$
0	$\boxed{1}$	$\boxed{\frac{3-1}{2-0} = 1}$	$\boxed{\frac{-1-1}{3-0} = -\frac{2}{3}}$	$\boxed{\frac{\frac{5}{6}+\frac{2}{3}}{5-0} = \frac{9}{30} = \frac{3}{10}}$
2	3	$\frac{2-3}{3-2} = -1$	$\frac{\frac{3}{2}+1}{5-2} = \frac{5}{6}$	
3	2	$\frac{5-2}{5-3} = \frac{3}{2}$		
5	5			

$$P(x) = f[x_0] + f[x_0, x_1](x-x_0) +$$
$$+ f[x_0, x_1, x_2](x-x_0)(x-x_1) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$$
$$= 1 + 1\cdot(x-0) - \frac{2}{3}\cdot(x-0)(x-2) + \frac{3}{10}(x-0)(x-2)(x-3)$$
$$= 1 + x - \frac{2}{3}x^2 + \frac{4}{3}x + \frac{3}{10}(x^3 - 5x^2 + 6x)$$
$$= \frac{3}{10}x^3 - \frac{13}{6}x^2 + \frac{62}{15}x + 1$$

príklad 3. S akou presnosťou sa dá vypočítať $\sqrt{115}$ pomocou Lagrangeovho interpolačného polynómu pre funkciu $f(x) = \sqrt{x}$, ak za uzly interpolácie zvolíme $x_0 = 100$, $x_1 = 121$, $x_2 = 144$?

$n = 2$, $x_0, x_1, x_2 \in [100, 144]$, $f \in C^3[100, 144]$

$f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$, $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$

príklad 3. pokračovanie

keďže $f^{(4)}(x) = -\frac{15}{16} x^{-\frac{7}{2}} < 0$ pre $x \in [100, 144]$

a zároveň $f'''(x) > 0$ pre $x \in [100, 144]$,

platí $\max\limits_{x \in [100,144]} |f'''(x)| = f'''(100) = \frac{3}{8} 100^{-\frac{5}{2}} =$

$= \frac{3}{8} \cdot \frac{1}{(\sqrt{100})^5} = \frac{3}{8 \cdot 10^5}$

$f'''(x)$ na $[100, 144]$ klesá

ak $\bar{x} \in [a, b]$, potom pre chybu interpolácie

platí $|f(\bar{x}) - P(\bar{x})| \leq \frac{|\omega_{n+1}(\bar{x})|}{(n+1)!} |f^{(n+1)}(\xi(\bar{x}))|$

$\leq \frac{|\omega_{n+1}(\bar{x})|}{(n+1)!} \max\limits_{x \in [a,b]} |f^{(n+1)}(x)|$

pre $\bar{x} = 115$ platí

$\omega_{n+1}(115) = (115 - 100)(115 - 121)(115 - 144)$

$= 15 \cdot (-6) \cdot (-29) = 6 \cdot 15 \cdot 29$

$\Rightarrow |\sqrt{115} - P(115)| \leq \frac{6 \cdot 15 \cdot 29}{3!} \cdot \frac{3}{8 \cdot 10^5} = \frac{45 \cdot 29}{8} \cdot 10^{-5}$

$= 163{,}125 \cdot 10^{-5} = 1{,}63125 \cdot 10^{-3}$.