$x_0, \dots, x_n$ uzly, $x_0 < \dots < x_n$

$r, d > 0$ prirodzené čísla, $S: [x_0, x_n] \to \mathbb{R}$

pre $x \in [x_k, x_{k+1}]$ ($k \in \{0, \dots, n-1\}$) je $S(x) = S_k(x)$,

kde S_k je polynóm stupňa r

S má spojité derivácie do rádu $r-d$ vrátane

r - stupeň, d - defekt

$\mathcal{S}_{r,d}$ - priestor splajnov stupňa r s defektom d

$\mathcal{S}_{1,1}$ - ~~po častiach spojité~~ po častiach lineárne spojité funkcie

$\mathcal{S}_{3,1}$ - po častiach polynóm stupňa 3 so spojitými deriváciami do rádu 2 (kubické splajny)

interpolačný splajn - má predpísané hodnoty $f_0, \dots, f_n$, teda $S(x_i) = f_i$

príklad 1. Pre uzly $0, 2, 3, 4, 6, 8$ a funkčné hodnoty $3, 1, 2, 0, -1, 1$ urči explicitne lineárny interpolačný splajn.

$$S_k(x) = a_k + b_k(x - x_k), \quad k \in \{0, \dots, n-1\}$$

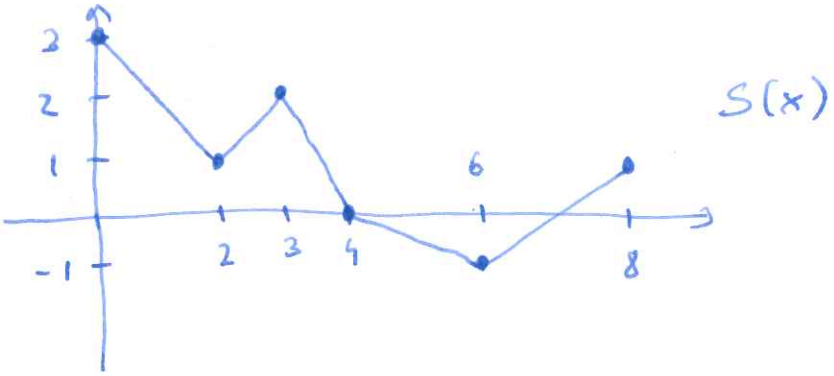
$a_k = f_k$

smernica: $b_k = \dfrac{f_{k+1} - f_k}{x_{k+1} - x_k}$

$$S(x) = \begin{cases} 3 + \frac{1-3}{2-0}(x-0) = -x+3, & x \in [0,2] \\ 1 + \frac{2-1}{3-2}(x-2) = x-1, & x \in [2,3] \\ 2 + \frac{0-2}{4-3}(x-3) = -2x+8, & x \in [3,4] \\ 0 + \frac{-1-0}{6-4}(x-4) = -\frac{x}{2}+2, & x \in [4,6] \end{cases}$$

príklad 1. pokračovanie

$$\left\{ -1 + \frac{1-(-1)}{8-6}(x-6) = x - 7, \qquad x \in [6,8] \right.$$



príklad 2. Pre uzly 0,1,2,3 a odpovedajúce funkčné hodnoty 1,-1,-3,1 nájdite kubický splajn. prirodzený

$$S_k = a_k + b_k(x-x_k) + c_k(x-x_k)^2 + d_k(x-x_k)^3, \quad k \in \{0,\dots,n-1\}$$

$$h_k = x_{k+1} - x_k, \quad k \in \{0,\dots,n-1\}$$

$$S_k' = b_k + 2c_k(x-x_k) + 3d_k(x-x_k)^2$$

$$S_k'' = 2c_k + 6d_k(x-x_k)$$

I. $S(x_k) = f_k, \quad k \in \{0,\dots,n\}$:

$$1 = f_0 = S_0(x_0) = a_0$$

$$-1 = f_1 = S_1(x_1) = a_1$$

$$-3 = f_2 = S_2(x_2) = a_2$$

$$\underline{1} = f_3 = S_2(x_3) = a_2 + b_2(x_3-x_2) + c_2(x_3-x_2)^2 + d_2(x_3-x_2)^2$$

$$\underline{= -3 + b_2 + c_2 + d_2}$$

II. S je spojitý v $x_1,\dots,x_{n-1}$: $S_k(x_{k+1}) = S_{k+1}(x_{k+1}),$
$k \in \{0,\dots,n-2\}$

$$S_0(x_1) = S_1(x_1)$$

$$\underline{1 + b_0 + c_0 + d_0 = -1}$$

$$S_1(x_2) = S_2(x_2)$$

$$\underline{-1 + b_1 + c_1 + d_1 = -3}$$

III. S' je spojitá v $x_1, \dots, x_{n-1}$: $S'_k(x_{k+1}) = S'_{k+1}(x_{k+1})$

$k \in \{0, \dots, n-2\}$

$$S'_0(x_1) = S'_1(x_1)$$

$$\underline{b_0 + 2c_0 + 3d_0 = b_1}$$

$$S'_1(x_2) = S'_2(x_2)$$

$$\underline{b_1 + 2c_1 + 3d_1 = b_2}$$

IV S'' je spojitá v $x_1, \dots, x_{n-1}$: $S''_k(x_{k+1}) = S''_{k+1}(x_{k+1})$

$k \in \{0, \dots, n-2\}$

$$S''_0(x_1) = S''_1(x_1)$$

$$2c_0 + 6d_0 = 2c_1 \Rightarrow c_0 + 3d_0 = c_1$$

$$S''_1(x_2) = S''_2(x_2)$$

$$2c_1 + 6d_1 = 2c_2 \Rightarrow c_1 + 3d_1 = c_2$$

V S je prirodzený splajn: $S''(x_0) = S''(x_n) = 0$

$$2c_0 = 0$$

$$2c_2 + 6d_2 = 0$$

z V a IV vyjadríme d: $d_2 = -\frac{c_2}{3}$

$$d_1 = \frac{c_2 - c_1}{3}$$

$$d_0 = \frac{c_1}{3}$$

rovnice z I, II a III tvoria systém 5 rovníc pre 5 neznámych:

a) $b_2 + c_2 - \frac{c_2}{3} = 4 \iff 3b_2 + 2c_2 = 12$

b) $b_0 + \frac{c_1}{3} = -2 \iff 3b_0 + c_1 = -6$

c) $b_1 + c_1 + \frac{c_2 - c_1}{3} = -2 \iff 3b_1 + 2c_1 + c_2 = -6$

d) $b_0 - b_1 + c_1 = 0$

e) $b_1 - b_2 + 2c_1 + c_2 - c_1 = 0 \iff b_1 - b_2 + c_1 + c_2 = 0$

príklad 2 - pokračovanie

$$
\begin{array}{ccccc|c}
b_0 & b_1 & b_2 & c_1 & c_2 & \\
\end{array}
$$

$$
\left(\begin{array}{ccccc|c}
0 & 0 & 3 & 0 & +2 & 12 \\
3 & 0 & 0 & 1 & 0 & -6 \\
0 & 3 & 0 & 2 & 1 & -6 \\
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & 0
\end{array}\right) \sim
$$

$$
\sim \left(\begin{array}{ccccc|c}
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 0 & 3 & 0 & +2 & 12 \\
0 & 3 & 0 & 2 & 1 & -6 \\
0 & 3 & 0 & -2 & 0 & -6
\end{array}\right) \sim \left(\begin{array}{ccccc|c}
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 0 & 3 & 0 & +2 & 12 \\
0 & 0 & 3 & -1 & -2 & -6 \\
0 & 0 & 3 & -5 & -3 & -6
\end{array}\right)
$$

$$
\sim \left(\begin{array}{ccccc|c}
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 0 & 3 & 0 & +2 & 12 \\
0 & 0 & 0 & -1 & -4 & -18 \\
0 & 0 & 0 & -5 & -5 & -18
\end{array}\right) \sim \left(\begin{array}{ccccc|c}
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 0 & 3 & 3 & +2 & 12 \\
0 & 0 & 0 & +1 & 4 & +18 \\
0 & 0 & 0 & 0 & -15 & -72
\end{array}\right)
$$

$$d_0 = \frac{c_1}{3} = -\frac{2}{5}$$

$$c_2 = \frac{72}{15} = \frac{24}{5}, \quad c_1 = 18 - 4\,\frac{24}{5} = \frac{90-96}{5} = -\frac{6}{5}$$

$$d_1 = \frac{c_2 - c_1}{3} = 2$$

$$b_2 = \frac{12 - \frac{48}{5}}{3} = 4 - \frac{48}{15} = \frac{60-48}{15} = \frac{12}{15} = \frac{4}{5}$$

$$b_1 = b_2 - c_1 - c_2 = \frac{12}{15} + \frac{6}{5} - \frac{24}{5} = \frac{4+6-24}{5} = -\frac{14}{5}$$

$$b_0 = b_1 - c_1 = -\frac{14}{5} + \frac{6}{5} = -\frac{8}{5}$$

$$d_2 = -\frac{c_2}{3} = -\frac{8}{5}$$

príklad 3. Uvažujme lineárnu diferenciálnu rovnicu $y' = f(x)\cdot y + g(x)$ s počiatočnou podmienkou $y(x_0) = y_0$ a uzly $x_0, \dots, x_n$, pričom $x_0 < \dots < x_n$. Nájdite postačujúce podmienky ~~p~~ na funkcie f a g, ktoré zaručia, že splajn typu $S_{2,1}$ existuje a spĺňa túto diferenciálnu rovnicu v uzloch $x_0, \dots, x_n$.

$I_k = [x_k, x_{k+1}]$

ak $x \in I_k$, potom $S(x) = S_k(x) = a_k + b_k(x-x_k) + c_k(x-x_k)^2$,

$k \in \{0, \dots, n-1\}$

I.	$S_k(x_{k+1}) = S_{k+1}(x_{k+1})$,	$k \in \{0, \dots, n-2\}$	$n-1$
II.	$S'_k(x_{k+1}) = S'_{k+1}(x_{k+1})$,	$k \in \{0, \dots, n-2\}$	$n-1$
III.	$S'(x_k) = f(x_k)S(x_k) + g(x_k)$,	$k \in \{0, \dots, n\}$	$n+1$
IV.	$S(x_0) = y_0$		1
			$3n$

$\Downarrow$

počet rovníc = počet parametrov

$S'_k(x) = b_k + 2c_k(x-x_k)$; $h_k = x_{k+1} - x_k$

$k \in \{0, \dots, n-1\}$

I. $a_{k+1} = a_k + b_k h_k + c_k h_k^2$ $\quad k \in \{0, \dots, n-2\}$

II. $b_{k+1} = b_k + 2c_k h_k$ $\quad k \in \{0, \dots, n-2\}$

III. $b_k = f(x_k)a_k + g(x_k)$ $\quad k \in \{0, \dots, n-1\}$

$b_{n-1} + 2c_{n-1}h_{n-1} = f(x_n)\left(a_{n-1} + b_{n-1}h_{n-1} + c_{n-1}h_{n-1}^2\right) + g(x_n)$

IV. $a_0 = y_0$

príklad 3. pokračovanie

pre $k \in \{0, \ldots, n-2\}$ platí:

$$c_k = \frac{b_{k+1} - b_k}{2h_k}, \quad b_k = f(x_k)a_k + g(x_k)$$
$$b_{k+1} = f(x_{k+1})a_{k+1} + g(x_{k+1})$$

$\Rightarrow$ z I dostávame pre $k \in \{0, \ldots, n-2\}$

$$a_{k+1} = a_k + (f(x_k)a_k + g(x_k))h_k + \frac{f(x_{k+1})a_{k+1} + g(x_{k+1}) - (f(x_k)a_k + g(x_k))}{2h_k} \cdot h_k^2$$

$\Leftrightarrow$

$$\cancel{\left(1 - \frac{f(x_{k+1})}{2h}\right)}$$

$$\left(1 - \frac{1}{2}f(x_{k+1})h_k\right)a_{k+1} + \left(-1 - \frac{f(x_k)}{2}h_k\right)a_k =$$
$$= g(x_k)h_k + \frac{g(x_{k+1}) - g(x_k)}{2}h_k$$

$a_0 = y_0$ poznáme a ak $1 - \frac{f(x_{k+1})h_k}{2} \neq 0$, potom môžeme dopočítať a_{k+1} pre $k \in \{0, \ldots, n-2\}$

keď už poznáme $a_0, \ldots, a_{n-1}$, môžeme z rovníc III dopočítať $b_0, \ldots, b_{n-1}$

keď už poznáme $b_0, \ldots, b_{n-1}$, môžeme z rovníc II dopočítať $c_0, \ldots, c_{n-2}$

z poslednej rovnice v III dostaneme

$$(2h_{n-1} - f(x_n)h_{n-1}^2)c_{n-1} = -b_{n-1} + f(x_n)(a_{n-1} + b_{n-1}h_{n-1}) + g(x_n)$$

takže ak $1 - \frac{f(x_n)h_{n-1}}{2} \neq 0$, môžeme dopočítať c_{n-1}.

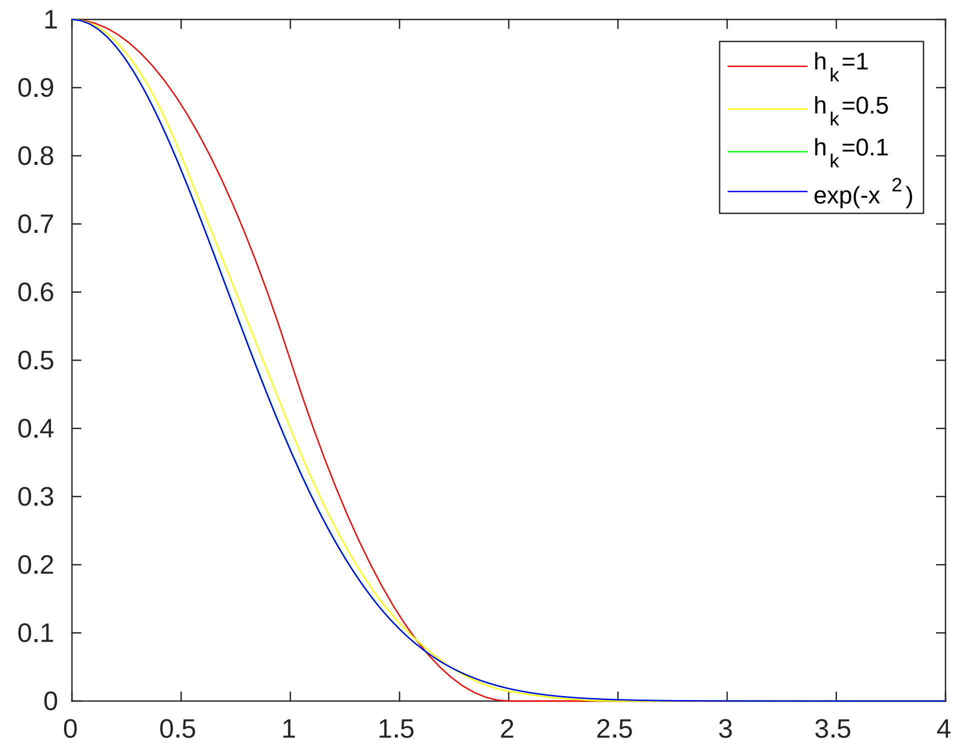
príklad 3. pokračovanie

– za predpokladu ohraničenosti (spojitosti) funkcie f sa nerovnosti $1 - \frac{f(x_{k+1}) h_k}{2} \neq 0$ pre $k \in \{0, \dots, n-1\}$ dajú zaručiť dostatočnou hustotou uzlov x_k, teda postačujúcou podmienkou je, že $\max_{k \in \{0, \dots, n-1\}} h_k$ je dostatočne malé.

– nerovnosti $1 - \frac{f(x_{k+1}) h_k}{2} \neq 0$ sú istè splnené v prípade, že $f < 0$ na intervale $[x_0, x_n]$.

---

tento postup môžeme použiť pre odhad funkčných hodnôt funkcie $f(x) = e^{-x^2}$, lebo tá je riešením počiatočného problému $y' = -2xy$, $y(0) = 1$



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3
3.5
4
h_k=1
h_k=0.5
h_k=0.1
exp(-x ^{2})