① Computing Tor

Recall $\mathrm{Tor}_n^R(A,B) := L_n(- \otimes_R B)(A) = H_n(P \otimes_R B)$ for a proj resol $P \to A[0]$.

1. Compute $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ by balancing.

Ans: By balancing, $L_n(- \otimes_R B)(A) \cong L_n(A \otimes -)(B)$.

Recall that we have a proj resol for $\mathbb{Z}/n\mathbb{Z}$:

$$\mathbb{Z} \xrightarrow{n\cdot -} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$$

right exact

Now apply $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} -$ to this resol (without $\mathbb{Z}/n\mathbb{Z}$):

$$\cdots 0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{n\cdot -} \mathbb{Z}/m\mathbb{Z} \to 0 \qquad \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \to 0$$

Then $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$

For $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$:

Note that if $(m,n)=1$, then

$$m \mid nx \iff m \mid x$$

so let $d = \gcd(m,n)$,

then $m \mid nx \iff \frac{m}{d} \mid \frac{n}{d}x$

$$\iff \frac{m}{d} \mid x$$

Thus, $\ker(n\cdot -) = \frac{m}{d}\mathbb{Z}/m\mathbb{Z}$

$$\cong \frac{m}{d}\mathbb{Z} / d\cdot\frac{m}{d}\mathbb{Z}$$

$$\cong \mathbb{Z}/d\mathbb{Z}$$

Then $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$.

2. Compute $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A)$ by relating to $\mathrm{Tor}_1(\mathbb{Q}, A)$.

Ans: Recall that $\mathbb{Z} \rightarrowtail \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$

Then considering the derived functors,

we obtain a LES

$$\cdots \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, A) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A) \xrightarrow{g} \mathbb{Z} \otimes_{\mathbb{Z}} A \xrightarrow{f} \mathbb{Q} \otimes_{\mathbb{Z}} A \to \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} A \to 0$$

$a \otimes x \qquad \frac{a}{1} \otimes x$

Now since $\mathbb{Q}$ is flat, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, A) \cong 0$ $\quad \because \mathbb{Q} \otimes_{\mathbb{Z}} -$ is exact

Thus $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A) \cong \mathrm{img}\, g = \ker f$

$$= TA, \text{ the torsion of } A$$

② Constructing the hom chain complex

Let $C, D$ be chain complexes, we construct a chain complex $\mathrm{Hom}(C, D)$ so as to give a closed sym monoidal structure on $Ch(Mod_R)$. In particular, the monoidal product should be symmetric:

$$B \otimes C \xrightarrow{\cong} C \otimes B$$
$$x \otimes y \mapsto (-1)^{|x| \cdot |y|} y \otimes x$$

We want to attain a bijection

$$\frac{B \otimes C \to D}{B \to \mathrm{Hom}(C, D)},$$

which, in terms of their underlying modules, is just

$$\frac{\sum_{n+k=l} B_n \otimes C_k \to D_l}{B_n \to \prod_k \mathrm{Hom}(C_k, D_{n+k})}$$

In other words, our job is to equip $\mathrm{Hom}(C,D)_n := \prod_k \mathrm{Hom}(C_k, D_{n+k})$ with a differential $D$ s.t. one map is a chain map $\Leftrightarrow$ the other map is.

Step I: Requiring the counit, which is the evaluation, to be a chain map,

$$ev: \mathrm{Hom}(C, D) \otimes C \to D$$
$$f \otimes c \mapsto f(c)$$

chain map def

by setting $d \cdot ev = ev \cdot (D \otimes 1 + 1 \otimes d)$

Step II: Apply $d \cdot ev$ to $f \otimes c$: recall $d^{\vee}(x \otimes y) = (-1)^{|x|} x \otimes dy$

$d \cdot ev(f \otimes c) = d(fc)$

$ev \cdot (D \otimes 1 + 1 \otimes d)(f \otimes c) = ev(Df \otimes c + (-1)^{|f|} f \otimes dc) = (Df)c + (-1)^{|f|} f(dc).$

So we can set $Df = df - (-1)^{|f|} fd = [d, f]$ graded commutator

Now $\mathrm{Hom}(C,D)$ has a differential, indeed

- The 0-chains are ordinary maps $f: C \to D$;
- The 0-cycles are those $Df = 0 \Leftrightarrow df = fd \Leftrightarrow$ chain maps (of deg 0)
- The $n$-cycles are those $Df = 0 \Leftrightarrow df = (-1)^n fd$ chain maps of deg $n$

Now for chain maps $f, g$ (0-cycles), we have

$$[f] = [g] \in H_0(\mathrm{Hom}(C,D)) \iff g - f \in B_0(\mathrm{Hom}(C,D))$$ $\hookrightarrow H_0 = Z_0 / B_0$

$$\iff \exists\, h \in \mathrm{Hom}(C,D)_1 : Dh = g - f$$

$$\iff dh + hd = g - f$$

$$\iff h \text{ is a chain htpy } f \simeq g.$$

Thus, we obtain $H_0(\mathrm{Hom}(C,D)) = [C,D]$, the chain htpy classes of chain maps $C \to D$.

This gives an alternative characterisation of chain htpies.