Recall that $\mathrm{Ext}^n_R(A,B) := R^n(\mathrm{Hom}_R(A,-))(B)$

- $\mathrm{Ext}^0_R(A,B) \cong \mathrm{Hom}_R(A,B)$
- B is inj R-mod $\Leftrightarrow \mathrm{Ext}^n_R(A,B) = 0 \quad \forall A,\ n \neq 0$
  $\Leftrightarrow \mathrm{Ext}^1_R(A,B) = 0 \quad \forall A$

1. Compute $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z})$ for a torsion group A.
Hint: $\mathbb{Q}$ is an injective $\mathbb{Z}$-mod.

Ans: SES giving an injective coresolution of $\mathbb{Z}$:

$$\mathbb{Z} \rightarrowtail \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$$

induces a LES when applied by the right derived functors:

$$0 \to \mathrm{Hom}(A,\mathbb{Z}) \xrightarrow{k} \mathrm{Hom}(A,\mathbb{Q}) \xrightarrow{h} \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}) \xrightarrow{g} \mathrm{Ext}^1(A,\mathbb{Z}) \xrightarrow{f} \mathrm{Ext}^1(A,\mathbb{Q}) \to \cdots$$

Since $\mathbb{Q}$ is injective, $\mathrm{Ext}^1(A,\mathbb{Q}) = 0$.
Now $\mathrm{Ext}^1(A,\mathbb{Z}) = \mathrm{coker}\ h$ $\qquad$ $\mathrm{coker} f = \mathrm{cod}/\mathrm{im} f$
Since A is torsion, $\mathrm{Hom}(A,\mathbb{Q}) = 0$, so

$$\mathrm{Ext}^1(A,\mathbb{Z}) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}).$$

Rk. If $A = \mathbb{Z}[\frac{1}{p}]$, i.e., adjoining the inverses of $a, a^2, \ldots$, then
$\mathrm{Hom}(A,\mathbb{Z}) = 0$: Note that $p \cdot \frac{1}{p} = 1$

$$p \cdot f(\tfrac{1}{p}) = f(1) \Rightarrow p \mid f(1)$$

Inductively, $p^n \mid f(1) \ \forall n \Rightarrow f(1) = 0$.

$$\therefore f \equiv 0.$$

$\mathrm{Hom}(A,\mathbb{Q}) = \mathbb{Q}$: $f(1)$ determines everything.
$\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_p \times \mathbb{Q}/\mathbb{Z}$: Firstly, there is $\mathbb{Q}/\mathbb{Z}$ choices of $f(1)$, as $\overline{f(1)} + k = \overline{f(1)}$.
Now for each fixed choice of $f(1)$, consider

$$p \cdot f(\tfrac{1}{p}) = f(1)$$

but this time we also have

$$p \cdot (f(\tfrac{1}{p}) + \tfrac{a}{p}) = f(1), \quad a \in \{0, \ldots, p-1\}$$

Altogether, we have p choices of $f(\frac{1}{p})$.
Similarly, p^n choices of $f(\frac{1}{p^k})$, and $f(\frac{1}{p^k})$ determines $f(\frac{1}{p^r})$, $r \leq k$.
So we have a diag $\qquad$ each choice of $f(\frac{1}{p})$

$$\cdots \to \mathrm{Hom}(\tfrac{1}{p^2}\cdot\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(\tfrac{1}{p}\cdot\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$$

And the answer is the inverse lim of this diag.
Evaluating, $\mathrm{Hom}(\frac{1}{p^n}\cdot\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$

$\therefore$ The lim is $\mathbb{Z}_p \times \mathbb{Q}/\mathbb{Z}$.

All in all, $\mathrm{Ext}^1(A,\mathbb{Z}) = (\mathbb{Z}_p \times \mathbb{Q}/\mathbb{Z}) / \mathbb{Q} \cong \mathbb{Z}_p/\mathbb{Z}$

Recall that

$\mathrm{cyl}\, C := \mathrm{Cyl}(\mathrm{id}_C)$

$$\mathrm{cyl}\, C_n = C_n \oplus C_{n-1} \oplus C_n, \quad d = \begin{pmatrix} +d^C & +\mathrm{id}_C & 0 \\ 0 & -d^C & 0 \\ 0 & -\mathrm{id}_C & +d^C \end{pmatrix}$$

As a double complex, $\mathrm{cyl}\, C = \mathrm{cyl}\, R[0] \otimes C$

with $\mathrm{cyl}\, R[0] = \cdots \to 0 \to R \xrightarrow{d} R \oplus R$

2. Describe concretely what is meant by a chain map $(f_-, h, f_+) : \mathrm{cyl}\, C \to D$.

Ans: A chain map satisfies

$$d^D \,(f_- \quad h \quad f_+) = (f_- \quad h \quad f_+) \begin{pmatrix} +d^C & +\mathrm{id}_C & 0 \\ 0 & -d^C & 0 \\ 0 & -\mathrm{id}_C & +d^C \end{pmatrix}$$

$$\Rightarrow \begin{cases} d^D \cdot f_- = f_- \cdot d^C \\ d^D \cdot h = f_- - h \cdot d^C - f_+ \Leftrightarrow f_- - f_+ = d^D h + h \cdot d^C \\ d^D \cdot f_+ = f_+ \cdot d^C \end{cases}$$

$\therefore$ f_+, f_- are chain maps $C \to D$ and h is a chain htpy $f_+ \sim f_-$.

Derivation & Principal derivation

A derivation of a ring R with coeff in an R-R-bimod M is a grp homo $D : R \to M$: $D(r \cdot s) = Dr \cdot s + r \cdot Ds$

A principal derivation is one of the form $p_x(r) = rx - xr$, $x \in M$

If $R = \mathbb{Z}G$, a left $\mathbb{Z}G$-mod M can be made into a right $\mathbb{Z}G$-mod via the trivial action, so $D(hg) = Dh + h \cdot Dg$.

3. Consider $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ and
$g \mapsto 1$

define a group homo $\delta : \ker \varepsilon \to M$
$(g-1) \mapsto D(g)$.

Prove that δ is a module homo, hence $\mathrm{Der}(\mathbb{Z}G, M) \cong \mathrm{Hom}_{\mathbb{Z}G}(\ker\varepsilon, M)$.

Pf.
$$D(hg) = Dh + h \cdot Dg$$
$$D(hg) - Dh = h \cdot Dg$$
$$\delta(hg) - \delta(h) = h \cdot \delta(g-1)$$
$$\delta(hg - h) = h \cdot \delta(g-1)$$
$$\therefore \delta(h(g-1)) = h \cdot \delta(g-1)$$
$\square$

4. Prove that $H^1(G; M) \cong \mathrm{Der}(\mathbb{Z}G, M) / \mathrm{PDer}(\mathbb{Z}G, M)$

Pf. Consider $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, $g \mapsto 1$, which is clearly surj,

so we have a SES
$$\ker\varepsilon \rightarrowtail \mathbb{Z}G \twoheadrightarrow \mathbb{Z},$$
which induces a LES when applied by the right derived functor:
$$0 \to \mathrm{Hom}(\mathbb{Z}, M) \to \mathrm{Hom}(\mathbb{Z}G, M) \to \mathrm{Hom}(\ker\varepsilon, M) \to \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, M) \to \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, A) \to 0$$
where $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, A) = 0$.

So
$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}(\mathbb{Z}, M) & \to & \mathrm{Hom}(\mathbb{Z}G, M) & \to & \mathrm{Hom}(\ker\varepsilon, M) & \to & \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, M) & \to 0 \\ & \wr\| & & \wr\| & & \| & & \| & \\ & M^G & \rightarrowtail & M & \to & \mathrm{Der}(\mathbb{Z}G, M) & \twoheadrightarrow & H^1(G; M) & \end{array}$$

Note that $M \to \mathrm{Hom}(\ker\varepsilon, M)$ spanned by ker
$m \mapsto \mu : (g-1) \mapsto (g-1) \cdot m = gm - m$,

since for any $\phi : \mathbb{Z}G \to M$, $\phi(g) = g \cdot \phi(1) =: g \cdot m$.

So μ corresponds to a principal der D_m.
$$\therefore H^1(G; M) \cong \mathrm{Der}(\mathbb{Z}G, M) / \mathrm{PDer}(\mathbb{Z}G, M).$$
$\square$