Recall that:

The n^{th} group homology with coefficients in a $\mathbb{Z}G$-mod M is

$$H_n(G;M) := L_n(-)_G(M)$$
$$= \mathrm{Tor}_n^{\mathbb{Z}G}(M,\mathbb{Z}) \text{ or } \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},M)$$

The n^{th} group cohomology with coefficients in a $\mathbb{Z}G$-mod M is

$$H^n(G;M) := R^n(-)^G(M)$$
$$= \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},M)$$

The (reduced) bar resolution

$$B_n = \mathbb{Z}G(G^n) \,/\, \{[g_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes g_n]\}$$

form a free (hence proj) resolution of $\mathbb{Z}$ in $\mathrm{Mod}_{\mathbb{Z}G}$, with $d_0[g_1|\cdots|g_n] = g_1\cdot[g_2|\cdots|g_n]$

$g_0 = 1$

$$d_i([g_1|\cdots|g_n]) = \begin{cases} [g_1|\cdots|g_i g_{i+1}|\cdots|g_n], & g_i g_{i+1} \neq 1 \\ 0, & g_i g_{i+1} = 1 \end{cases}$$

$$d_n[g_1|\cdots|g_n] = [g_1|\cdots|g_{n-1}]$$

$$d := \sum_i (-1)^i d_i$$

Thm 10.14

Let G be a finite group of order k. The mult by k is 0 on $H_n(G;M)$ and $H^n(G;M)$ for $n>0$.

Pf. We show that $k\cdot-$ on B is htpic to the map

$$N = \begin{cases} 0, & n>0 \\ \sum_{g\in G} g\cdot -, & n=0 \end{cases}$$

Define $h[g_1|\cdots|g_n] = (-1)^{n+1}\cdot\sum_{g\in G}[g_1|\cdots|g_n|g]$.

i) Compute $(d_i h + h d_i)[g_1|\cdots|g_n]$, $n\neq 0$:

Ans: $d_i h[g_1|\cdots|g_n] + h d_i[g_1|\cdots|g_n]$

$$= d_i\,(-1)^{n+1}\cdot\sum_g [g_1|\cdots|g_n|g] + h[g_1|\cdots|g_i g_{i+1}|\cdots g_n]$$

$$= (-1)^{n+1}\sum_g [g_1|\cdots|g_i g_{i+1}|\cdots|g_n|g] + (-1)^n\cdot\sum_g [g_1|\cdots|g_i g_{i+1}|\cdots|g_n|g]$$

$$= 0$$

ii) Compute $d^{n+1}h + h d^n\ [g_1|\cdots|g_n]$, $n\neq 0$:

Ans: $d^{n+1}h + h d^n$

$$= \sum_{n+1}(-1)^{n+1} d_{n+1} h + h\sum_n (-1)^n d_n$$

$$= (-1)^{n+1} d_{n+1} h \qquad \text{by (i)}$$

Now $(-1)^{n+1} d_{n+1} h[g_1|\cdots|g_n] = (-1)^{n+1}\cdot(-1)^{n+1}\sum_g d_{n+1}[g_1|\cdots|g_n|g]$

$$= \sum_g [g_1|\cdots|g_n] = k\cdot[g_1|\cdots|g_n].$$

$\therefore\ d^{n+1}h + h d^n = k$ for $n\neq 0$.

iii) Compute $d^1 h + h d^0 [\,]$, i.e., $n=0$:

Ans: $d^1 h + h d^0 = d_0 h - d_1 h + h d_0$

Now $d_0 h[\,] - d_1 h[\,] + h \underbrace{d_0 [\,]}_{=0}$

$= d_0 h[\,] - d_1 h[\,]$

$= - d_0 \sum_g [g] + d_1 \sum_g [g]$

$= - \sum_g g \cdot [\,] + \sum_g [\,]$

$= -N[\,] + k[\,]$.

So $h: N \sim k$.

Applying $M \otimes_{\mathbb{Z}G} -$ and $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ to h, we obtain chain htpies

$M \otimes_{\mathbb{Z}G} N \sim M \otimes_{\mathbb{Z}G} k : M \otimes_{\mathbb{Z}G} B \to M \otimes_{\mathbb{Z}G} B$

and $\mathrm{Hom}_{\mathbb{Z}G}(N, M) \sim \mathrm{Hom}(k, M) : \mathrm{Hom}_{\mathbb{Z}G}(B, M) \to \mathrm{Hom}_{\mathbb{Z}G}(B, M)$:

$$\begin{array}{ccccccc}
\cdots & M \otimes_{\mathbb{Z}G} B_2 & \longrightarrow & M \otimes_{\mathbb{Z}G} B_1 & \longrightarrow & M \otimes_{\mathbb{Z}G} B_0 \\
 & 0 \downarrow \sim \downarrow M\otimes k & & 0 \downarrow \sim \downarrow M\otimes k & & M\otimes N \downarrow \sim \downarrow M\otimes k \\
\cdots & M \otimes_{\mathbb{Z}G} B_2 & \longrightarrow & M \otimes_{\mathbb{Z}G} B_1 & \longrightarrow & M \otimes_{\mathbb{Z}G} B_0
\end{array}$$

Similarly for $\mathrm{Hom}_{\mathbb{Z}G}(B, M)$.

Now $f \sim g \Rightarrow H_*(f) = H_*(g)$, $H^*(f) = H^*(g)$. □

Coroll 10.15

Let G and M be finite with $(|G|, |M|) = 1$. Then $H_n(G; M) = 0$ and $H^n(G; M) = 0$ for $n > 0$.

Pf. The mult by $k = |G|$ is 0 for $n > 0$ by Thm 10.14, in other words, $H_*(G; M)$ & $H^*(G; M)$ are $\mathbb{Z}/|G|\mathbb{Z}$ - modules.

Now since $(|G|, |M|) = 1$, by Bezout's Identity,

$a|G| + b|M| = 1$.

Now $akm = m - blm = m$ since $\mathrm{ord}(m) \mid l$ by Lagrange's Thm.

Thus $M \xrightarrow{k\cdot} M$ is an iso.

$\therefore H_n(G; M) \underset{\cong}{\xrightarrow{0}} H_n(G; M)$, $H^n(G; M) \underset{\cong}{\xrightarrow{0}} H^n(G; M)$. □

Recall Infinite cyclic gp C_∞ with generator t

$\mathbb{Z}C_\infty$ is the ring of Laurent poly

A proj resol of $\mathbb{Z}$ is given by

$$\cdots \to 0 \to \mathbb{Z}C_\infty \xrightarrow{t-1} \mathbb{Z}C_\infty \xrightarrow{ev_1} \mathbb{Z}$$

1. Compute the homology of C_∞ with coefficient $\mathbb{Z}$.

Ans: i) Apply $- \otimes_{\mathbb{Z}C_\infty} \mathbb{Z}$,

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

tensoring = ev

$t \otimes n = 1 \otimes tn = 1 \otimes n$

$\mathbb{Z}C_\infty$-action on $\mathbb{Z}$ is trivial

ii) Take homology,

$$H_n(C_\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0,1 \\ 0, & n=2,3,4\ldots \end{cases}$$

2. Compute the cohomology of C_∞ with coefficient $\mathbb{Z}$.

Ans: i) Apply $\mathrm{Hom}_{\mathbb{Z}C_\infty}(-,\mathbb{Z})$,

$$\cdots \leftarrow 0 \leftarrow \underset{\cong\,\mathbb{Z}}{\mathrm{Hom}_{\mathbb{Z}C_\infty}(\mathbb{Z}C_\infty,\mathbb{Z})} \xleftarrow{0} \underset{\cong\,\mathbb{Z}}{\mathrm{Hom}_{\mathbb{Z}C_\infty}(\mathbb{Z}C_\infty,\mathbb{Z})}$$

module hom

$f(t) = f(t\cdot 1) = t\cdot f(1) = f(1)$

ii) Take cohomology,

$$H^n(C_\infty;\mathbb{Z}) = \begin{cases} \ker 0 = \mathrm{Hom}(\mathbb{Z}C_\infty,\mathbb{Z}) \cong \mathbb{Z}, & n=0 \\ \mathrm{coker}\, 0 = \mathrm{Hom}(\mathbb{Z}C_\infty,\mathbb{Z}) \cong \mathbb{Z}, & n=1 \\ 0, & n=2,3,4,\ldots \end{cases}$$