Recall that a rep of G over $\Bbbk$ is a $\Bbbk G$-mod V, equivalently, is a $\Bbbk$-vector space V with a group homo $G \to GL(V)$. ~ group action

The dual of V is $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$.

1. Suppose char $\Bbbk \mid |G|$, show that the $\Bbbk G$-mod $\Bbbk G$ is injective, i.e., the group algebra $\Bbbk G$ is self-injective.

Ans: Claim: $(\Bbbk G)^*$, the dual of $\Bbbk G$, is an injective $\Bbbk G$-mod:
Consider for a monic $X \rightarrowtail Y$,

$$\begin{tikzcd} X \otimes_{\Bbbk G} \Bbbk G \arrow[r, "\Bbbk\text{-lin}"] \arrow[d, tail, "\Bbbk\text{-lin}"'] & \Bbbk \\ Y \otimes_{\Bbbk G} \Bbbk G \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\Bbbk G$-mod

Since $\Bbbk G$ is flat over $\Bbbk G$, $X \otimes_{\Bbbk G} \Bbbk G \rightarrowtail Y \otimes_{\Bbbk G} \Bbbk G$ is monic.
Since $\Bbbk$ is injective $\exists$ $Y \otimes_{\Bbbk G} \Bbbk G \to \Bbbk$ making the triangle comm.
Applying the tensor-hom adj:
$$\mathrm{Hom}_S(Y \otimes_R X, Z) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z))$$

$$\begin{tikzcd} X \arrow[r, "\Bbbk G\text{-lin}"] \arrow[d, tail, "\Bbbk G\text{-lin}"'] & \mathrm{Hom}_{\Bbbk}(\Bbbk G, \Bbbk) = \Bbbk G^* \\ Y \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$(f \cdot r)(x) := f(r \cdot x)$

$\therefore$ $\Bbbk G^*$ is inj.

Claim: $\Bbbk G^* \cong \Bbbk G$:
Define $\Bbbk G \xrightarrow{\Bbbk\text{-lin}} \Bbbk G^*$
$g \mapsto \delta_{g^{-1}}$
We can extend $\delta_{g^{-1}} : G \to \Bbbk$ to $\Bbbk G \to \Bbbk$ by setting
$$\delta_{g^{-1}}(kh) := k\, \delta_{g^{-1}}(h)$$
So for $\sum a_g \cdot g \in \Bbbk G$,
we have $\sum a_g \cdot \delta_{g^{-1}} \in \Bbbk G^*$, that is defined via
$$\left(\sum a_g \cdot \delta_{g^{-1}}\right)\left(\sum b_h \cdot h\right) = \left(\sum_g a_g \cdot \delta_{g^{-1}}\right)(b_{h_1} \cdot h_1) + \left(\sum_g a_g \cdot \delta_{g^{-1}}\right)(b_{h_2} \cdot h_2) + \cdots$$
$$= \sum_h \left(\sum_g a_g \cdot \delta_{g^{-1}}\right)(b_h \cdot h) = \sum_{h = g^{-1}} a_g \cdot b_h$$
Now define $\Bbbk G^* \longrightarrow \Bbbk G$
$$\zeta : \Bbbk G \to \Bbbk \quad \mapsto \quad \sum_g \zeta(g^{-1}) \cdot g$$
We have $\sum_g \left(\sum_h a_h \cdot \delta_{h^{-1}}\right)(g^{-1}) \cdot g = \sum_g a_g \cdot g$ and
$$\left(\sum_g \zeta(g^{-1}) \cdot \delta_{g^{-1}}\right)\left(\sum_h a_h \cdot h\right) = \sum_{h = g^{-1}} \zeta(g^{-1}) \cdot a_h = \sum_h a_h \zeta(h) = \zeta\left(\sum_h a_h \cdot h\right).$$

Recall that the dihedral group D_8 consists of symmetries of a square
i.e., $D_8 = \langle r, s : r^4 = s^2 = 1, srs^{-1} = r^{-1} \rangle$

A rep is irreducible $\Leftrightarrow$ its only quotient (equiv. subspaces) are V and 0 (equiv. 0 and V).
In other words, no other proper G-invariant subspaces.

2. Find all the irr representations of D_8 over $\mathbb{C}$.

Ans: In $\mathbb{C}$, the reflection across a line l has invariant subspaces 0, l, $l^\perp$, $\mathbb{C}^2$.
In D_8, we have reflections across the line $x=0$ & $x=y$, with common invariant subspaces 0, $\mathbb{C}^2$.

First, a trivial D_8-action is clearly a 1-dim irr rep.
Next, consider $r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,
and $y=0$ is the invariant subspace under action by s,
and $y=0$ is not invariant under action by r because
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}.$$
$\therefore$ Irreducible.

Now since $8 = |G| = 2^2 + 1^2 + \sum \dim|V_i|^2 \Rightarrow 3 = \sum \dim |V_i|^2$,
the only irr left are all 1-dim.
More explicitly, they are given by

$r \mapsto 1$, $s \mapsto -1$
$r \mapsto -1$, $s \mapsto 1$
$r \mapsto -1$, $s \mapsto -1$.