Gröbner basis

Motivations : - decide whether a poly $f \in I = (g_1, \dots, g_s)$
- decide whether $(g_1, \dots, g_s) = (h_1, \dots, h_t)$

In 1-var case, we can solve by computing f/g and inspect the remainder.
This requires an ordering of terms by powers.
We first fix an order for the multi-variable case:

$x > y$

Def. (lexicographical order) x^0y^2 v.s. x^1y^0

We say $x_1^{\alpha_1} \cdots x_n^{\alpha_n} > x_1^{\beta_1} \cdots x_n^{\beta_n}$ $x^1y^0 > x^0y^2$

$\Leftrightarrow$ for some $i \geq 1$, $\alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i$.

Notations. When $f = a_\alpha x^\alpha + \sum_{\beta < \alpha} a_\beta x^\beta$ where $a_\alpha \neq 0$,
we call a_α the leading coeff & write $LCf = a_\alpha$;
x^α the leading monomial & write $LMf = x^\alpha$;
$a_\alpha x^\alpha$ the leading term & write $LTf = a_\alpha x^\alpha$.

Rk. this is a total order.

Def. Let $I \subseteq \Bbbk[X]$ be an ideal in a (multi-variable) poly ring.
We say that $g_1, \dots, g_s \in I$ form a Gröbner basis for I $\Leftrightarrow$
$\forall g \in I$, LMg is divisible by LMg_i for some g_i,
i.e., $(LMg) = (LMg_1, \dots, LMg_s)$.

Buchberger algorithm
Let $I = (f_1, \dots, f_\ell)$.

Step I. For each $f_i, f_j \in I$, find $x^\alpha = \mathrm{lcm}(LTf_i, LTf_j)$
Compute $S(f_i, f_j) := \frac{x^\alpha}{LTf_i} f_i - \frac{x^\alpha}{LTf_j} f_j$

Step II. Subtract monomial multiples of f_k's from $S(f_i, f_j)$ in Step I to cancel as many leading terms as possible. Warning: S is red by $g \Leftrightarrow LTg \mid$ term of S

Step III. If the end product $\bar{S}(f_i, f_j) \neq 0$, set $f_{\ell+n} := \bar{S}(f_i, f_j)$ and include it into $\{f_1, \dots, f_\ell\}$.

Step IV. Repeat Step I ~ Step III with the new collection $\{f_1, \dots, f_{\ell+n}\}$, until no new member can be added into this collection.

terminate: well-founded

Def. A Gröbner basis $\{g_1, \dots, g_s\}$ is reduced $\Leftrightarrow$ for any g_i, $LC\, g_i = 1$, and no term of g_i is divisible by any $LM\, g_j$ for all $j \neq i$.

Rk. This is like the reduced row echelon form of a matrix.

Turning a GB into a reduced GB
Let H be a collection, initially empty.

Step I. Replace each g_i in the Gröbner basis by $\frac{g_i}{LC\, g_i}$, take all these to form a collection F.

Step II. Pick $g_i \in F$.
If for all g_k in the collection $F \setminus \{g_i\}$, $LT\, g_k \nmid LT\, g_i$, and for all $h \in H$, $LT\, h \nmid LT\, g_i$,
then add g_i into H.

Step III. Repeat Step II until all $g_i \in F$ are picked.

Step IV. For each $h \in H$, compute $\bar{h}$ by subtracting monomial multiples of $H \setminus \{h\}$ to cancel as many leading terms as possible, then replace h with $\bar{h} \neq 0$.
If $\bar{h} = 0$, discard h. Warning: S is red by $g \Leftrightarrow LT\, g \mid$ term of S

Step V. Repeat Step IV until all h are replaced / discarded.

1. Compute the reduced Gröbner basis of $I = (f_1, f_2)$, where $f_1 = x^3 - 2xy$, $f_2 = x^2y + x - 2y^2$, w.r.t to $x > y$
priority

Ans: $S(f_1, f_2) = \frac{x^3y}{x^3}(x^3 - 2xy) - \frac{x^3y}{x^2y}(x^2y + x - 2y^2)$
$= -x^2$

Set $f_3 := -x^2$, add f_3 into the collection.

$S(f_1, f_3) = \frac{x^3}{x^3}(x^3 - 2xy) - \frac{x^3}{x^2}(x^2) = -2xy =: f_4$

$S(f_2, f_3) = \frac{x^2y}{x^2y}(x^2y + x - 2y^2) - \frac{x^2y}{x^2}(x^2) = x - 2y^2 =: f_5$

$S(f_1, f_4) = \frac{-2x^3y}{x^3}(x^3 - 2xy) - \frac{-2x^3y}{-2xy}(-2xy)$
$= -2y(x^3 - 2xy) - x^2(-2xy)$
$\to 4xy^2 + 2y f_4 = 0$

$S(f_2, f_4) = \frac{-2x^2y}{x^2y}(x^2y + x - 2y^2) - \frac{-2x^2y}{-2xy}(-2xy)$
$= -2x^2y - 2x + 4y^2 + 2x^2y$
$\to -2x + 4y^2 + 2f_5 = 0$

Similarly $S(f_3, f_4)$, $S(f_1, f_5)$, $S(f_2, f_5)$, $S(f_3, f_5) \to 0$

$S(f_4, f_5) = \frac{-2xy}{-2xy}(-2xy) - \frac{-2xy}{x}(x - 2y^2)$
$= -2xy + 2xy - 4y^3$
$= -4y^3 =: f_6$

$\therefore \{f_1, f_2, f_3, f_4, f_5, f_6\}$ is GB.

Now replace the collection with

$F = \{g_1 = x^3 - 2xy,\ g_2 = x^2y + x - 2y^2,\ g_3 = x^2,\ g_4 = xy,\ g_5 = x - 2y^2,\ g_6 = y^3\}$

Pick g_1:
$LTg_1 = x^3$, but $LTg_3 \mid x^3$.

Pick g_2:
$LT g_2 = x^2y$, but $LTg_3 \mid x^2y$

Pick g_3:
$LTg_3 = x^2$, but $LTg_5 \mid x^2$

Pick g_4:
$LT g_4 = xy$, but $LTg_5 \mid xy$

Pick g_5: No LTg_k divides LTg_5
$\Rightarrow g_5 \in H$.

Pick g_6: No LTg_k or LTh divides LT
$\Rightarrow g_6 \in H$

$\therefore \{g_5, g_6\}$ is a reduced GB.

2. Solve $\begin{cases} x^3 - 2xy = 0 \\ x^2y + x - 2y^2 = 0 \end{cases}$

By Q1., this is equivalent to ask

$$\begin{cases} x - 2y^2 = 0 \\ y = 0 \end{cases}$$

$$\Leftrightarrow \quad x = y = 0.$$

3. Find the reduced GB of $I = (f_1, f_2, f_3)$, where

$f_1 = x^2+y^2+z^2-1$, $f_2 = x^2 - y + z^2$, $f_3 = x - z$., $x > y > z$

$$S(f_1, f_2) = x^2+y^2+z^2-1 - (x^2-y+z^2)$$
$$= y^2 + y - 1 =: f_4$$

$$S(f_1, f_3) = x^2+y^2+z^2-1 - x(x-z)$$
$$\to xz + y^2 + z^2 - 1 \quad - z f_3 \quad - f_4$$
$$= z^2 - 1 + z^2 - y + 1$$
$$= -y + 2z^2 =: f_5$$

$$S(f_1, f_4) = \frac{x^2y^2}{x^2}(x^2+y^2+z^2-1) - \frac{x^2y^2}{y^2}(y^2+y-1)$$
$$= y^2(x^2+y^2+z^2-1) - x^2(y^2+y-1)$$
$$y^4 + y^2z^2 - y^2 - x^2y + x^2$$
$$\to x^2(-y+1) + y^2(y^2+z^2-1) \quad + (y-1) f_1$$
$$= \cancel{x^2(-y+1)} + y^2(y^2+z^2-1) + (y-1)(\cancel{x^2} + y^2+z^2-1)$$
$$\to (y^2+y-1)(y^2+z^2-1) \quad - (y^2+z^2-1) f_4$$
$$= 0$$

$$S(f_2, f_3) = x^2 - y + z^2 - x(x-z)$$
$$\to xz - y + z^2 \quad - z f_3$$
$$\to -y + 2z^2 \quad - f_5 \quad = 0$$

$$S(f_3, f_5) = \frac{-xy}{x}(x-z) - \frac{-xy}{-y}(-y+2z^2)$$
$$= -y(x-z) - x(-y+2z^2)$$
$$\to -2xz^2 + yz \quad + z f_5$$
$$\to -2xz^2 + 2z^3 \quad + 2z^2 f_3 \quad = 0$$

$$S(f_2, f_5) = \frac{-x^2y}{x^2}(x^2-y+z^2) - \frac{-x^2y}{-y}(-y+2z^2)$$
$$= -y(x^2-y+z^2) - x^2(-y+2z^2)$$
$$\to y^2 - yz^2 - 2x^2z^2 \quad - f_4 \quad - z^2 f_5$$
$$= \cancel{y^2} - \cancel{yz^2} - 2x^2z^2 - \cancel{y^2} - y + 1 + \cancel{yz^2} - 2z^4$$
$$\to -2x^2z^2 - y - 2z^4 + 1 \quad - f_5 \quad + 2xz^2 f_3$$
$$= \cancel{-2x^2z^2} \cancel{-y} - 2z^4 + 1 \cancel{+y} - 2z^2 + \cancel{2x^2z^2} - 2xz^3$$
$$\to -2xz^3 - 2z^4 - 2z^2 + 1 \quad + 2z^3 f_3$$
$$= \cancel{-2xz^3} - 2z^4 - 2z^2 + 1 + \cancel{2xz^3} - 2z^4$$
$$= -4z^4 - 2z^2 + 1 =: f_6$$

Similarly, $S(f_1, f_5)$, $S(f_1, f_6)$, $S(f_2, f_6)$, $S(f_3, f_6)$, $S(f_4, f_6)$,
$S(f_5, f_6)$, $S(f_2, f_4)$, $S(f_3, f_4)$, $S(f_4, f_6) \to 0$

$\therefore \{f_1, f_2, f_3, f_4, f_5, f_6\}$ is a GB.

Now set

$g_1 = x^2+y^2+z^2-1$, $g_2 = x^2-y+z^2$, $g_3 = x-z$, $g_4 = y^2+y-1$, $g_5 = y-2z^2$, $g_6 = z^4+\frac{1}{2}z^2-\frac{1}{4}$.

Pick g_1:

$LT\, g_1 = x^2$ but $LT\, g_2 \mid x^2$

g_2:

$LT\, g_2 = x^2$ but $LT\, g_3 \mid x^2$

g_3:

$LT\, g_3 = x$

$\Rightarrow g_3 \in H$

g_4:

$LT\, g_4 = y^2$ but $LT\, g_5 \mid y^2$

g_5:

$LT\, g_5 = y$

$\Rightarrow g_5 \in H$

g_6:

$LT\, g_6 = z^4$

$\Rightarrow g_6 \in H$

$\therefore \{g_3, g_5, g_6\}$ is the reduced GB.