Application of GB — finding the intersection of 2 ideals

Let $I = (f_i)$, $J = (g_j)$.

Introduce a new variable t, $t > x_k$, and consider the collection

$$\{ tf_1, \ldots tf_r, (1-t)g_1, \ldots, (1-t)g_s \}$$

and denote by K the ideal gen by this collection.

A poly in K can be written as

$$tf + (1-t)g = t(f-g) + g,$$

the term $t(f-g)$ vanishes precisely when $f = g$ which is exactly that $g \in I \cap J$.

Now compute the Gröbner basis B_t for the ideal K.

Dump all the poly that contain multiples of t in their terms in B_t, and obtain a collection B.

Then $(B) = I \cap J$.

1. Consider the ring $\mathbb{k}[x,y]$, $I = (x, x^2y^2, y^3)$, $J = (x^2, y^2)$. Find $I \cap J$.

Hint: $t > x > y$.

$$-tx^2 + x^2 \qquad -ty^2 + y^2$$

Ans: $K = (tx, tx^2y^2, ty^3, (1-t)x^2, (1-t)y^2)$

$$S(k_1, k_2) = \frac{tx^2y^2}{tx}(tx) - \frac{tx^2y^2}{tx^2y^2}(tx^2y^2)$$
$$= tx^2y^2 - tx^2y^2 = 0$$

Similarly, $S(k_1, k_3) = S(k_2, k_3) = 0$

$$S(k_1, k_4) = \frac{tx^2}{tx}(tx) - \frac{tx^2}{tx^2}(-tx^2 + x^2)$$
$$= -x^2 =: k_6$$

$$S(k_1, k_5) = \frac{txy^2}{tx}(tx) - \frac{txy^2}{ty^2}(-ty^2 + y^2)$$
$$= -xy^2 =: k_7$$

$$S(k_3, k_5) = \frac{ty^3}{ty^3}(ty^3) - \frac{ty^3}{ty^2}(-ty^2 + y^2)$$
$$= -y^3 =: k_8$$

$$S(k_4, k_5) = \frac{tx^2y^2}{tx^2}(-tx^2 + x^2) - \frac{tx^2y^2}{ty^2}(-ty^2 + y^2)$$
$$= x^2y^2 - x^2y^2 = 0$$

So $B_t = \{k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8\}$

$B = \{k_6, k_7, k_8\}$

$\therefore \quad I \cap J = (x^2, xy^2, y^3)$

Localisation

Idea: To add some (multiplicative) inverses into a ring.

Motivation: In Algebraic Geometry, we consider rational functions $\frac{f}{g}$ defined at a point p, they are precisely the invertible (as $g(p) \neq 0$, given $f(p) \neq 0$). These functions form a local ring.

We consider commutative ring with 1.

Def. A local ring is a ring with a unique maximal ideal.

E.g. Any field is a local ring. The unique max ideal is (0).

Def. Let A be a ring and $D \subseteq A$ be a multiplicative subset, i.e., $1 \in D$ and, $x, y \in D \Rightarrow xy \in D$.

Consider the equiv relation $\sim$ on $A \times D$ as follows:

$$(a_1, d_1) \sim (a_2, d_2) \iff \exists\, d \in D : (a_1 d_2 - a_2 d_1) d = 0.$$

d can be not 1

The quotient $D^{-1}A := A \times D / \sim$ is the localisation of A w.r.t. D, its class is denoted $[a, d] = \frac{a}{d}$.

A ring structure on $D^{-1}A$ is defined with

$$\frac{a_1}{d_1} + \frac{a_2}{d_2} = \frac{a_1 d_2 + a_2 d_1}{d_1 d_2}, \quad \frac{a_1}{d_1} \cdot \frac{a_2}{d_2} = \frac{a_1 a_2}{d_1 d_2}.$$

There is a map $\lambda : A \to D^{-1}A$, $a \mapsto \frac{a}{1}$, which is a ring homo.

Rk. $D^{-1}A$ enjoys the uni property:

Let $\rho : A \to B$ be a ring homo s.t. $\rho(d) \in B^{\times}$ is a unit $\forall\, d \in D$. There exists a unique ring homo $\tilde{\rho} : D^{-1}A \to B$ s.t. $\rho = \tilde{\rho}\lambda$:

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\lambda"'] & B \\ D^{-1}A \arrow[ur, dashed, "\exists!\,\tilde{\rho}"'] & \end{tikzcd}$$

1. Let R be a commutative ring with 1.

Let $U = \{1, a, a^2, \cdots\} \subseteq R$.

Show that $U^{-1}R \cong R[x]/(ax-1)$

Hint: use uni prop:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\lambda"'] & S \\ U^{-1}R \arrow[ur, dashed, "\exists! t"'] & \end{tikzcd}$$

Ans: We have a ring homo $f : R \to S$ s.t. $f(u) \in S^\times \ \forall u \in U$, so in particular, $f(a) \in S^\times$.

There is a unique evaluation homo

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \\ R[x] \arrow[ur] & \end{tikzcd}$$

$$\bar{f} : R[x] \to S$$
$$x \mapsto f(a)^{-1}$$

and we have $\bar{f}(ax-1) = f(a)f(a)^{-1} - 1 = 0$,

so $\bar{f}$ induces a unique ring homo

$$t : R[x]/(ax-1) \to S$$

s.t. $t \circ \lambda = f$.

Rk. In this case, we denote $U^{-1}R$ by $R[a^{-1}]$ or $R[\frac{1}{a}]$.