2. Show that if $P \subseteq R$ is a prime ideal, then $R \setminus P$ is a multiplicative set.

Ans: P is prime: $ab \in P \Rightarrow a \in P$ or $b \in P$.

Now suppose $a \in R \setminus P$ and $b \in R \setminus P$.

If $ab \notin R \setminus P$, then $ab \in P$. But if $ab \in P$, then $a \in P$ or $b \in P$, which means a & b cannot both be in $R \setminus P$.

So $ab \in R \setminus P$.

Def. Let $P \subseteq A$ be a prime ideal in a ring A.

The localisation of A w.r.t. $A \setminus P$ is denoted by A_P, it is called the localisation of A at P.

3. Compute $\mathbb{Z}_{(0)}$ and $\mathbb{Z}_{(p)}$ for a prime no. $p \in \mathbb{Z}$.

Ans: $\mathbb{Z} \setminus (0) = \mathbb{Z} \setminus \{0\}$. So $\mathbb{Z}_{(0)} \cong \mathbb{Q}$.

$\mathbb{Z} \setminus (p) = \mathbb{Z} \setminus \{np : n \in \mathbb{Z}\}$.

So $\mathbb{Z}_{(p)} \cong \{ \frac{a}{b} : b \neq np\} = \{\frac{a}{b} : p \nmid b\}$ is a subring of $\mathbb{Q}$.

Def. An R-mod N is flat $\Leftrightarrow$ $(-)\otimes_R N : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor, i.e., preserves SES.

Def. A category C is called filtered $\Leftrightarrow$
- it is non-empty;
- for any two obj c_1, c_2, $\exists\, c_3 \in C$ and mor $c_1 \to c_3$, $c_2 \to c_3$;
- for any two parallel mor $f, g : c_1 \rightrightarrows c_2$, $\exists\, h : c_2 \to c_3$ s.t. $hf = hg$

$$\begin{tikzcd}
c_1 \arrow[dr] & \\
 & c_3 \\
c_2 \arrow[ur] &
\end{tikzcd}$$

generalisation of directed sets

$$\begin{tikzcd}
c_1 \arrow[r, shift left] \arrow[r, shift right] & c_2 \arrow[r] & c_3
\end{tikzcd}$$

Rk. A module is flat $\Leftrightarrow$ it is a filtered colim of free modules.

4. Show that $U^{-1}R$ is flat as an R-mod.
Hint: use the above Rk. $\frac{r}{v^k} = \frac{r u^k}{(uv)^k}$
And note that $R[\frac{1}{v}] \hookrightarrow R[\frac{1}{uv}]$ corresponds to multiplication by u $R \xrightarrow{u\cdot} R$, where $u, v \in U$.

Ans: $U^{-1}R$ can be built as the colim of the diagram

$$\begin{tikzcd}
 & & R \arrow[r, dotted] & \cdots \\
 & R \arrow[ur, "h\cdot"] \arrow[dr, "v\cdot"] & & \cdots \\
R \arrow[ur, "u\cdot"] \arrow[dr, "v\cdot"'] & & R \arrow[ur, "h\cdot"] \arrow[dr, "k\cdot"] & \\
 & R \arrow[ur, "u\cdot"] \arrow[dr, "k\cdot"'] & & R \arrow[r, dotted] & \cdots \\
 & & R \arrow[ur, "u\cdot"'] & \\
 & & & \cdots
\end{tikzcd}$$

and this is clearly a filtered diagram.

5. Let M be a fin gen Ab group.
Compute $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$.

Ans: Write $M \cong (\mathbb{Z}^r \oplus \mathbb{Z}/(n_1) \oplus \cdots \oplus \mathbb{Z}/(n_r))$

Then $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$

$$\cong \underbrace{\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}}_{r \text{ copies}} \oplus \mathbb{Z}_{(p)} \otimes \mathbb{Z}/(n_1) \oplus \cdots \oplus \mathbb{Z}_{(p)} \otimes \mathbb{Z}/(n_r)$$

Note that $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}_{(p)}$
and $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$
$\cong \mathbb{Z}/(p^k)$ for $n = p^k j \neq 0$

So $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$
$\cong \mathbb{Z}_{(p)}{}^r \oplus (\mathbb{Z}/(p^{k_1}) \oplus \cdots \oplus \mathbb{Z}/(p^{k_r}))$
where $n_r = p^{k_r} \cdot j_r$ where $p \nmid j_r$.

Rk. $\frac{a}{b} \otimes x$, $x \in \mathbb{Z}/(n)$, $n = p^k j \neq 0$
$= \frac{a}{jb} \otimes jx$
$= 0$ where $p^k \mid x$

$(\mathbb{Z}/n\mathbb{Z}) / \underbrace{(p^k)(\mathbb{Z}/n\mathbb{Z})}_{\cong \mathbb{Z}/j\mathbb{Z}} \cong \mathbb{Z}/(p^k)$

In particular,
$\frac{a}{b} \otimes x$, $x \in \mathbb{Z}/(n)$, $p \nmid n$ $\quad k=0$
$= \frac{a}{bn} \otimes nx$
$= \frac{a}{bn} \otimes 0 = 0$

$\frac{a}{b} \otimes x$, $x \in \mathbb{Z}/(p)$ $\quad k=1, j=1$
$= \frac{1}{b} \otimes ax$
$= \frac{1}{(b \cdot \bar{b})} \otimes \bar{b}ax$ where $\bar{b}b = b\bar{b} = 1 \pmod p$
$= 1 \otimes a\bar{b}x \cong a\bar{b}x$