

2. Show that if  $P \subseteq R$  is a prime ideal, then  $R \setminus P$  is a multiplicative set.

Ans:  $P$  is prime:  $ab \in P \Rightarrow a \in P$  or  $b \in P$ .

Now suppose  $a \in R \setminus P$  and  $b \in R \setminus P$ .

If  $ab \notin R \setminus P$ , then  $ab \in P$ . But if  $ab \in P$ , then  $a \in P$  or  $b \in P$ , which means  $a$  &  $b$  cannot both be in  $R \setminus P$ .

So  $ab \in R \setminus P$ .

Def. Let  $P \subseteq A$  be a prime ideal in a ring  $A$ . The localisation of  $A$  w.r.t.  $A \setminus P$  is denoted by  $A_P$ , it is called the localisation of  $A$  at  $P$ .

3. Compute  $\mathbb{Z}_{(0)}$  and  $\mathbb{Z}_{(p)}$  for a prime no.  $p \in \mathbb{Z}$ .

Ans:  $\mathbb{Z} \setminus (0) = \mathbb{Z} \setminus \{0\}$ . So  $\mathbb{Z}_{(0)} \cong \mathbb{Q}$ .

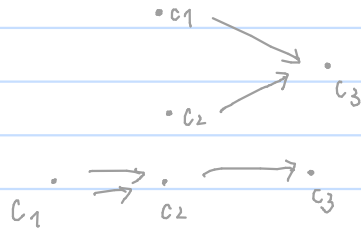
$\mathbb{Z} \setminus (p) = \mathbb{Z} \setminus \{np : n \in \mathbb{Z}\}$ ,

So  $\mathbb{Z}_{(p)} \cong \left\{ \frac{a}{b} : b \neq np \right\} = \left\{ \frac{a}{b} : p \nmid b \right\}$   
is a subring of  $\mathbb{Q}$ .

Def. An  $R$ -mod  $N$  is flat  $\Leftrightarrow (-) \otimes_R N : \text{Mod}_R \rightarrow \text{Mod}_R$  is an exact functor, i.e., preserves SES.

Def. A category  $\mathcal{C}$  is called filtered  $\Leftrightarrow$

- it is non-empty;
- for any two obj  $c_1, c_2$ ,  $\exists c_3 \in \mathcal{C}$  and mor  $c_1 \rightarrow c_3, c_2 \rightarrow c_3$ ;
- for any two parallel mor  $f, g: c_1 \rightarrow c_2$ ,  $\exists h: c_2 \rightarrow c_3$  s.t.  $hf = hg$



Generalisation of directed sets

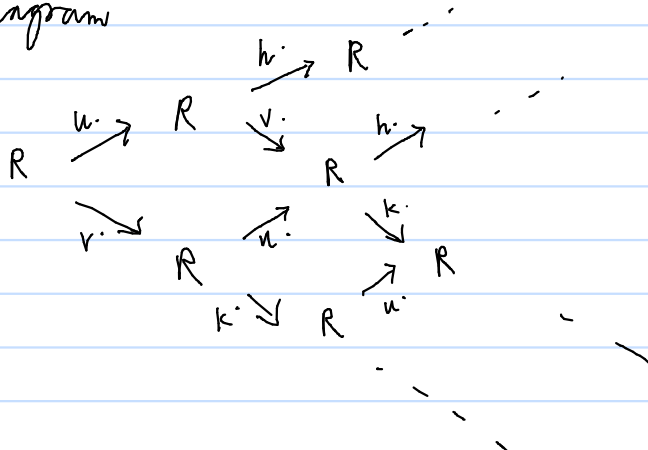
Rk. A module is flat  $\Leftrightarrow$  it is a filtered colim of free modules.

4. Show that  $U^{-1}R$  is flat as an  $R$ -mod.

Hint: use the above Rk.

And note that  $R[\frac{1}{v}] \hookrightarrow R[\frac{1}{uv}]$  corresponds to multiplication by  $u$   $R \xrightarrow{u} R$ , where  $u, v \in U$ .

Ans:  $U^{-1}R$  can be built as the colim of the diagram



and this is clearly a filtered diagram.

5. Let  $M$  be a fin gen Ab group.  
 Compute  $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$ .

Ans: Write  $M \cong (\mathbb{Z}^r \oplus \mathbb{Z}/(n_1) \oplus \dots \oplus \mathbb{Z}/(n_r))$

then  $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$   
 $\cong \underbrace{\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z} \oplus \dots \oplus \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}}_{r \text{ copies}} \oplus \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}/(n_1) \oplus \dots \oplus \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}/(n_r)$

Note that  $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}_p$   
 and  $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathbb{Z}/(p^k)$  for  $n = p^k j \neq 0$

So  $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$   
 $\cong \mathbb{Z}_p^r \oplus (\mathbb{Z}/(p^{k_1}) \oplus \dots \oplus \mathbb{Z}/(p^{k_r}))$   
 where  $n_r = p^{k_r} \cdot j_r$  where  $p \nmid j_r$ .

Rk.  $\frac{a}{b} \otimes x$ ,  $x \in \mathbb{Z}/(n)$ ,  $n = p^k j \neq 0$   
 $= \frac{a}{j \cdot b} \otimes jx$   
 $= 0$  where  $p^k \mid x$   
 $(\mathbb{Z}/n\mathbb{Z}) / \underbrace{(\frac{p^k}{j} \mathbb{Z}/n\mathbb{Z})}_{\cong \mathbb{Z}/j\mathbb{Z}} \cong \mathbb{Z}/(p^k)$

In particular,  
 $\frac{a}{b} \otimes x$ ,  $x \in \mathbb{Z}/(n)$ ,  $p \nmid n$   $k=0$   
 $= \frac{a}{bn} \otimes nx$   
 $= \frac{a}{bn} \otimes 0 = 0$

$\frac{a}{b} \otimes x$ ,  $x \in \mathbb{Z}/(p)$   $k=1, j=1$   
 $= \frac{1}{b} \otimes ax$   
 $= \frac{1}{(b \cdot \bar{b})} \otimes \bar{b}ax$  where  $\bar{b}b = b\bar{b} = 1 \pmod{p}$   
 $= 1 \otimes a\bar{b}x \cong a\bar{b}x$