1. a) An ideal P is prime $\Leftrightarrow$ for any ideals I, J,
$IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.

Pf. $(\Rightarrow)$: Suppose $I \subsetneq P$. Then $\exists\, x \in I : x \notin P$.
Now $\forall y \in J$, $xy \in IJ \subseteq P$
$\Rightarrow y \in P$ as P is prime.
So $J \subseteq P$.

$(\Leftarrow)$: Suppose $xy \in P$.
In particular, pick $I := (x)$, $J := (y)$, so $IJ \subseteq P$.
Now by assumption, $(x) \subseteq P$ or $(y) \subseteq P$.
Hence $x \in P$ or $y \in P$.

b) If P is prime, then for any ideals I, J,
$I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.

Pf. Suppose $I \subsetneq P$. Then $\exists\, x \in I : x \notin P$.
Suppose that $I \subsetneq P$ and $J \subsetneq P$.
i.e., $\exists\, x \in I : x \notin P$ and $\exists\, y \in J : y \notin P$.
Now since P is prime $xy \notin P$,
and we know that $xy \in I \cap J$.
$\therefore I \cap J \subsetneq P$.

c) How about the converse of (b)?

Ans: False. Consider the ideal $(4) \subseteq \mathbb{Z}$.
Any ideal lying in (4) is of the form $(4k)$ for $k \in \mathbb{N}$,
so $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$. Yet (4) is not prime.

Def. A minimal prime ideal M is a prime ideal containing 0 such that if there is another prime ideal Q containing 0, then $M \subseteq Q$.

2. For any prime ideal P, there exists a minimal prime ideal M with $M \subseteq P$.
Hint: The intersection of a descending chain of prime ideals is a prime ideal, and apply Zorn's Lemma.

Pf. Consider the collection
$$S = \{ Q : Q \text{ is prime } \& \ Q \subseteq P \}$$
Clearly, $P \in S$, so $S \neq \phi$.
Now consider a descending chain $\{Q_i\}$ in S, ordered by inclusion. Since $\cap Q_i$ is also a prime ideal, $\cap Q_i \in S$. By Zorn's Lemma, there is a minimal element.

Rk. Suppose $xy \in \cap Q_i$ and $x, y \notin \cap Q_i$.
Then $\exists\, i, j : x \notin Q_i, \ y \notin Q_j$.
WLOG, $Q_i \subseteq Q_j$.
So $x, y \notin Q_i$.
But $xy \in Q_i$, contradiction.

Def. An Ab-cat is a cat enriched in Ab,
i.e., the set of morphisms $A \to B$ has an Ab grp structure.
In particular, the composition is distributive:

$$f \circ (g+h) = (f \circ g) + (f \circ h), \quad f: B \to C, \quad g, h: A \to B$$

$$(f+g) \circ h = (f \circ h) + (g \circ h), \quad f, g: B \to C, \quad h: A \to B.$$

Def. An Ab-cat is an Abelian cat precisely when
- it has a zero obj;
  i.e., an obj 0 st $\forall X$, $\exists!\ 0 \to X$
  and also $\forall X$, $\exists!\ X \to 0$.
- it has all binary biproducts;
  i.e., both a prod & coprod
- it has all kernels & cokernels;
  i.e.,

$$\ker f \rightarrowtail X \underset{0}{\overset{f}{\rightrightarrows}} Y, \qquad X \underset{0}{\overset{f}{\rightrightarrows}} Y \twoheadrightarrow \operatorname{coker} f$$

- all monomorphisms & epimorphisms are (co)normal;
  i.e., it is the ker (resp. coker) of some mor.

E.g. Mod_R, $\mathrm{Sh}_{Ab}(X)$ - cat of sheaves on X of Ab.