1. Show that in an Abelian cat, the pullback of an epi f is also an epi.

i.e., $h\bar{f} = 0 \Rightarrow h = 0$.

Hint: 1. Show that $P = \ker(f, -g)$, where $(f,-g) : A \oplus C \xrightarrow{f-g} B$
2. Show that there is a SES
$$P \overset{i}{\rightarrowtail} A \oplus C \twoheadrightarrow B$$
3. Use $\bar{f} = \pi_C \cdot i$, where $\pi_C : A \oplus C \twoheadrightarrow C$
4. Use $(f, -g) \cdot \iota_A = f$, where $\iota_A : A \rightarrowtail A \oplus C$

Ans: Consider

$$\begin{tikzcd} P \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ A \arrow[r, two heads, "f"'] & B \end{tikzcd}$$

Consider $(f, -g) : A \oplus C \to B$
$a \oplus c \mapsto fa - gc$.

Since f is an epi, i.e., $hf = kf \Rightarrow h = k$,
so $h \circ (f, -g) = k \circ (f, -g) \Rightarrow (hf, -hg) = (kf, -kg)$
$\Rightarrow h = k$ which means $(f,-g)$ is an epi.

Indeed, $a \oplus c \in \ker(f, -g) \Leftrightarrow fa = gc \Leftrightarrow (a, c) \in P$.
$\therefore P = \ker(f, -g)$

Now we have a SES
$$0 \to P = \ker(f,-g) \overset{i}{\rightarrowtail} A \oplus C \xrightarrow{(f,-g)} B \to 0.$$

Let π_C denote the proj $A \oplus C \to C$.
Then $\bar{f} = \pi_C \cdot i$.

Suppose $h\bar{f} = 0$, we want to show that $h = 0$.
We have $h \cdot \pi_C \cdot i = 0$

$$\begin{tikzcd} P \arrow[r, "i"] & A \oplus C \arrow[r, "\pi_C"] \arrow[dr, two heads, "(f,-g)"'] & C \arrow[r, "h"] & X \\ & & B \arrow[ur, "x"'] & \end{tikzcd}$$

This means $\ker(f, -g) = \operatorname{im} i \subseteq \ker(h \cdot \pi_C)$,
so $h \cdot \pi_C$ factors through B,
i.e., $x \cdot (f, -g) = h \cdot \pi_C$

Let ι_A denote the inc $A \to A \oplus C$.
Then $x \cdot (f, -g) \cdot \iota_A = h \cdot \pi_C \cdot \iota_A = 0$
$\Rightarrow x \cdot f = 0$
$\Rightarrow x = 0$

$$A \xrightarrow{\iota_A} A \oplus C \xrightarrow{(f,-g)} B \xrightarrow{x} X$$
$$a \mapsto a \oplus 0 \mapsto f(a) \mapsto x \cdot f(a)$$

$\therefore h \cdot \pi_C = 0 \Rightarrow h = 0$ as π_C is epi.

Def. (Projective resolution)
A resolution of a mod A is a non-negatively graded chain complex C with an aumentation map $\varepsilon : C_0 \to A$ s.t.

$$\cdots \to C_1 \to C_0 \xrightarrow{\varepsilon} A$$

is an acyclic chain complex, i.e., the augmented chain complex is an LES.
C is a projective resol $\Leftrightarrow$ C consists of proj mod

2. Construct a projective resolution of $\mathbb{Z}/n\mathbb{Z}$ in $\mathrm{Mod}_{\mathbb{Z}}$.

Ans: We always have the following SES

$$0 \to \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$$

this gives a proj res of $\mathbb{Z}/n\mathbb{Z}$.

3. Let $C_2 = \{1, t\}$ be the cyclic group of 2 elements, i.e., $t^2 = 1$.
Denote by $\mathbb{Z}C_2$ the group ring of C_2, consisting of elements of the form $a + b\cdot t$ with mult given by

$$(a+b\cdot t)(c+dt) = (ac+bd) + (ad+bc)t$$

$$ac + adt + bct + bd$$

for $a, b, c, d \in \mathbb{Z}$.
Construct a proj res of $\mathbb{Z}$ in $\mathrm{Mod}_{\mathbb{Z}[C_2]}$.

Hint: Consider $\mathbb{Z}[C_2] \xrightarrow{t-1} \mathbb{Z}[C_2]$ and $\mathbb{Z}[C_2] \xrightarrow{t+1} \mathbb{Z}[C_2]$.

$$1 \mapsto t-1 \qquad\qquad 1 \mapsto t+1$$

$$t \mapsto t(t-1) = 1-t \qquad\qquad t \mapsto t(t+1) = t+1$$

Ans: It is clear that $\mathbb{Z}[C_2] \xrightarrow{ev_1} \mathbb{Z}$ is surjective.
Now consider

$$\cdots \to \mathbb{Z}[C_2] \xrightarrow{t+1} \mathbb{Z}[C_2] \xrightarrow{t-1} \mathbb{Z}[C_2] \xrightarrow{ev_1} \mathbb{Z}$$

Since $(t-1)(t+1) = 0$, so $\cdots \to \mathbb{Z}[C_2]$ is a chain complex.
It remains to show the exactness at $\mathbb{Z}[C_2]$'s.

$$\begin{aligned}(a+bt)(t+1) &= a+b + (a+b)t \\ &= (a+b)(t+1)\end{aligned} \qquad\qquad \begin{aligned}(a+bt)(t-1) &= (-a+b) + (a-b)t \\ &= (a-b)(t-1)\end{aligned}$$

So $\mathrm{im}\,(t+1) = \mathbb{Z}\cdot(t+1)$, $\mathrm{im}\,(t-1) = \mathbb{Z}\cdot(t-1)$

$\ker(t+1) = \{a+bt : a+b=0\} = \{(-c+d) + (c-d)t\} = \mathbb{Z}\cdot(t-1)$

$\ker(t-1) = \{a+bt : a-b=0\} = \{(c+d) + (c+d)t\} = \mathbb{Z}\cdot(t+1)$

Hence we are done.