Higher cats course

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Course has 2 parts:

Part 1 (L1-6)

- In L1-6, we study globular ∞-groupoids, focusing on the (globular theory)-based approach and Grothendieck ∞-groupoids.

We cover (in detail):

- cats & groupoids as models of "graphical theories".
- Globular sets & globular theories
- Grothendieck ∞-groupoids
- Fundamental ∞-groupoids & the homotopy hypothesis.
- Construction of globular ∞-groupoids associated to identity types.

Note: Along the way, we give a careful construction of the "efficient small object argument" & explain how it captures free constructions in universal algebra (L4).

Part 2 (L7-11)

Quick overview of simplicial models of higher cat, covering quasicategories, complete Segal spaces, some models of $(\infty,2)$-cats & higher + ∞-cosmoi.

Lecture 1 —
Groupoids & categories revisited

- What are the theories of categories & groupoids & what makes them special (amongst other theories)?
- What do we mean by a "theory"?
- Well, for classical algebraic structures, one can answer this question with Lawvere theories.
- In this setting we are interested in sets X with operations

$$X^{n} \xrightarrow{m_X} X$$

arities are natural numbers

satisfying some equations.

- We want to view operations as maps $1 \xrightarrow[m]{} n$ in a cat $\mathbb{T}$ & our algebra X as a functor

$$\begin{tikzcd} \mathbb{T}^{op} \arrow[r, "X"] & \mathrm{Set} \\ m \arrow[d] & X^n \arrow[d, "m_X"] \\ n \arrow[r, mapsto] & X^1 = X \end{tikzcd}$$

- For this reason, we take our cat. of arities $\mathbb{N}$ to be the cat of fin. ordinals $n = \{0, \dots, n-1\}$ for $n \in \mathbb{N}$ & functions between them.

This category has some canonical coproducts

$$\begin{tikzcd} 1 \arrow[dr, "i_0"'] & 1 \arrow[d, "i_1"] \cdots & 1 \arrow[dl, "i_{n-1}"] \\ & n & \end{tikzcd}$$

where i_j picks out the element $j \in 1$.

- A Lawvere theory is an identity on objects

$$J : \mathbb{N} \longrightarrow \mathbb{T}$$

functor preserving these coproducts (equally all finite coproducts) & a model of $\mathbb{T}$ is a functor

$$X : \mathbb{T}^{op} \longrightarrow \mathrm{Set}$$ sending these coproducts to products

$$\begin{tikzcd} 1 \arrow[dr] & & & X(1) \\ 1 \arrow[r] & n \arrow[r, mapsto] & X(n) \cong X(1)^n \arrow[ur] \arrow[r] \arrow[dr] & \vdots \\ \vdots \ 1 \arrow[ur] & & & X(1) \end{tikzcd}$$

$\mathrm{Mod}(\mathbb{T}) \subseteq (\mathbb{T}, \mathrm{Set})$ is full subcategory of the functor cat. containing the $\mathbb{T}$-models.

Ex. - In the Lawvere theory $\mathbb{T}$ for monoids, we have a map $1 \xrightarrow{m} 2$, which gets sent to $X(1)^2 \cong X(2) \xrightarrow{X(m)} X(1)$

a binary op.

- In general, given a type of algebraic structure $T = (\Omega, \mathbf{E})$, we calculate the associated Lawvere theory $\mathbb{T}$ as follows: consider the adjunction

$$\begin{tikzcd} \mathrm{Alg}(T) \arrow[r, shift right, "U" below] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right, "F" above] \end{tikzcd}$$

F - free algebra

U - Forgetful

we define $\mathbb{T}$ by factoring

$$\begin{tikzcd} \mathbb{N} \arrow[r, hook, "\mathrm{inc}"] \arrow[dr, "I" above, "\text{identity on objects}" below left] & \mathrm{Set} \arrow[r, "F"] & \mathrm{Alg}(T) \\ & \mathbb{T} \arrow[ur, "J" above, "\text{Fully Faithful}" below right] & \end{tikzcd}$$

so $\mathbb{T}(n,m) = \mathrm{Alg}(T)(\mathbf{F}n, Fm)$.

with composition as in $\mathrm{Alg}(T)$.

E.g. in the case of monoids

$1 \xrightarrow{m} 2 \in \Pi$ corresponds to monoid map

$$\begin{array}{ccc} F1 & \longrightarrow & F2 \\ \parallel & & \parallel \\ \mathbb{N} & \longrightarrow & \text{Words}\{a,b\} \\ 1 & \longmapsto & [a,b] = [a].[b] \end{array}.$$

In this ~~setting~~, we always have

$$\begin{tikzcd} \mathrm{Alg}(T) \arrow[rr, equal, "\text{equiv}"] \arrow[dr, "U^T"'] & & \mathrm{Mod}(\Pi) \arrow[dl, "U^\Pi"] & X \arrow[d, mapsto] \\ & \mathrm{Set} & & X(1) \end{tikzcd}$$

so we can treat classical algebraic structures (involving operations

$$X^n \longrightarrow X \quad)$$

using Lawvere Theories.

- But what about categories & groupoids?

A category X is not a set but a directed graph

$$X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0$$

& involves operations like

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \longrightarrow & X_1 \\ \| & & \| \\ \mathrm{Graph}(0 \to 1 \to 2, X) & \longrightarrow & \mathrm{Graph}(0 \to 1, X) \end{array}$$

arities

- As such, we define our category $\Delta_0 \subseteq \mathrm{Graph}$ of arities to consist of the graphs

$$[n] = \{0 \to 1 \to 2 \to \cdots \to n\}$$

for $n \geq 0$. no endomorphisms!

- In Δ_0, we have the maps

$$[0] \underset{1}{\overset{0}{\rightrightarrows}} [1]$$

picking out 0 & 1 of $[1]$,

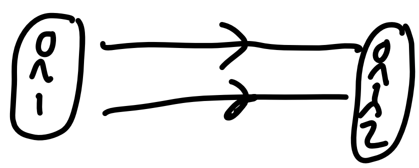
but there are not many maps:

For instance no maps

$$\begin{array}{ccc}[2] & \longrightarrow & [1]\\ \| & & \| \\ \boxed{0\to1\to2} & \longrightarrow & \boxed{1\to2}\end{array}$$

- The only maps in Δ_0 are the <u>distance preserving embeddings</u> $[n] \longrightarrow [m]$ for $n < m$.

E.g.



- Each $[n]$ is canonically a <u>wide pushout</u>

$$\begin{tikzcd}[column sep=small]
& {[0]} \arrow[dl, "0"'] \arrow[dr, "1"] & & {[0]} \arrow[dl, "0"'] \arrow[dr] & & & {[0]} \arrow[dl, "0"'] \arrow[dr, "1"] & \\
{[1]} \arrow[drrr, "i_1"'] & & {[1]} \arrow[dr, "i_2"] & & \cdots & {[1]} \arrow[dl] & & {[1]} \arrow[dlll, "i_n"] \\
& & & {[n]} & & & &
\end{tikzcd}$$

& we'll call <u>these</u> wide pushouts <u>graphical sums</u>.

- A graphical theory consists of an identity on obs functor
$$J : \Delta_0 \longrightarrow \Pi$$
preserving these graphical sums.
- A model of Π is a functor
$$X : \Pi^{op} \longrightarrow Set$$
sending graphical sums in Π to graphical products (wide pullbacks)
this just says that the induced map
$$X[n] \longrightarrow \overbrace{X[1] \times_{X[0]} X[1] \times \cdots \times_{X[0]} X[1]}^{n \text{ copies}}$$
is invertible & is called the Segal condition.
- $Mod(\Pi) \subseteq [\Pi^{op}, Set]$ is full subcat of presheaves sat. Segal condition.

The theory of categories

We calculate the theory Π_{cat} of categories by factoring

$$\begin{tikzcd} \Delta_0 \arrow[r, hook] \arrow[dr, "I"', "\text{id on obs}"] & \mathrm{Graph} \arrow[r, "\text{Free } F"] & \mathrm{Cat} \\ & \Pi_{cat} \arrow[ur, "J", "\text{ff.}"'] & \end{tikzcd}$$

as before.

What does it look like?

Obs : $[n] = \{0 \to 1 \to \dots \to n\}$.

Morphisms: $F[n] \longrightarrow F[m] \in \mathrm{Cat}$?

Well the free cat on $[n]$ is just $[n]$ viewed as an ordinal (or cat) with all composites & ids added –

Thus $\Pi_{cat} = \Delta$, the simplicial category of finite non-empty ordinals & order-preserving maps between them.

& $\Pi_{cat} = \Delta \xhookrightarrow[J]{} \mathrm{Cat}$ the full inclusion.

- $\Delta_0 \xhookrightarrow{I} \Delta$ is the obvious id. on obs functor & pres. glob. sums –

This is the graphical theory of categories.

For instance, $\begin{bmatrix} 0 \\ \downarrow \\ 1 \end{bmatrix} \rightrightarrows \begin{bmatrix} 0 \\ \downarrow \\ 1 \\ \downarrow \\ 2 \end{bmatrix}$ in Δ (not Δ_0)

corresponds to

map $X_1 \times_{X_0} X_1 \xrightarrow{\text{comp}} X_1$ in a cat X.

The so-called nerve functor

$$N = \mathrm{Cat}(J-,1) : \mathrm{Cat} \longrightarrow [\Delta^{op}, \mathrm{Set}]$$

sends

$$C \longmapsto NC \quad \text{where}$$

$$NC(n) = \mathrm{Cat}([n], C) =$$

$\{$ composable sequences $a_0 \xrightarrow{f_0} a_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n$ in C $\}$

It is fully faithful & has in its ess. image those simplicial sets sat. the Segal condition.

this is Grothendieck's nerve theorem.

This says that

$$\mathrm{Cat} \xrightarrow{\simeq} \mathrm{Mod}(\Delta) \hookrightarrow [\Delta^{op}, \mathrm{Set}]$$

so categories $\equiv$ models of Δ.

So indeed we can capture categories using graphical theories.

What about groupoids?

Factoring
$$\begin{tikzcd} \Delta_0 \arrow[r, hook] \arrow[dr, "I"'] & \mathrm{Grph} \arrow[r, "F"] & \mathrm{Gpd} \\ & \Pi_{\mathrm{Gpd}} \arrow[ur, "J"'] & \end{tikzcd}$$

as before, what is a map

$$F[n] \to F[m] \in \mathrm{Gpd}\ ?$$

Well $F[n] = F\{0 \to 1 \to 2 \to \cdots \to n\}$

$= \{0 \rightleftarrows 1 \rightleftarrows 2 \cdots n\text{-}1 \rightleftarrows n\}$ is in fact contractible: non-empty & $\exists!$ iso between any 2 obs.

Because of this, a functor

$F[n] \longrightarrow F[m]$ is uniquely specified by the function between sets of objects – ie. a function $[n] \to [m]$ (not nec. ord pres.)

So $\Delta_0 \longrightarrow \Pi_{\mathrm{Gpd}} = \mathbb{F}$ - finite non-empty ordinals & functions.

- For instance $\begin{bmatrix} 0 \\ \downarrow \\ 1 \end{bmatrix} \rightleftarrows \begin{bmatrix} 1 \\ \downarrow \\ 0 \end{bmatrix}$ encodes

$$\begin{tikzcd} X_1 \arrow[rr, "\mathrm{inv}"] \arrow[dr, shift right, "s"'] \arrow[dr, shift left, "t"] & & X_1 \arrow[dl, shift right, "t"'] \arrow[dl, shift left, "s"] \\ & X_0 & \end{tikzcd}$$

in a groupoid.

- The inclusion $\mathbb{F} \overset{J}{\hookrightarrow} \mathrm{Gpd}$ induces the symmetric nerve functor

$$\mathrm{Gpd} \longrightarrow [\mathbb{F}^{op}, \mathrm{Set}]$$

which restricts to an equivalence

$$\mathrm{Gpd} \overset{\sim}{\hookrightarrow} \mathrm{Mod}(\mathbb{F})$$

with those presheaves satisfying the Segal condition.

(This is the symmetric nerve theorem.)

What makes the theory of groupoids special?

- Consider $\Delta_0 \xrightarrow{J} \Pi_{Gpd} = \mathbb{F}$.

recalling $\Pi_{Gpd}([n],[m]) = Gpd(F[n], F[m])$

where
$$\begin{tikzcd} Gpd \arrow[r, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & Gph \arrow[l, shift right=1.5, "F"'] \end{tikzcd}$$

- Recall that given $x, y \in UF[n]$, $\exists!\, x \to y$.

i.e. given
$$\begin{tikzcd} {[0]} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & UF[n] \\ {[1]} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

or equivalently,
$$\begin{tikzcd} F[0] \arrow[r, shift left, "\bar{f}"] \arrow[r, shift right, "\bar{g}"'] \arrow[d, shift right, "I0"'] \arrow[d, shift left, "I1"] & F[n] \\ F[1] \arrow[ur, "\exists!\, h"'] & \end{tikzcd}$$

Def^n) A graphical theory $I : \Delta_0 \to \mathbb{T}$ is contractible

if given
$$\begin{tikzcd} {[0]} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "I0"'] \arrow[d, shift left, "I1"] & {[n]} \in \mathbb{T} \\ {[1]} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Example) By the above, the theory of groupoids is contractible.

- In fact, any contractible theory $\mathbb{T}$ encodes groupoids:

e.g. have

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "0"] \arrow[r, shift right, "2"'] & {[2]} \\ {[1]} \arrow[u, shift left, "0"] \arrow[u, shift right, "1"'] \arrow[ur, "\exists ! c"'] & \end{tikzcd} \quad \& \quad \begin{tikzcd} {[0]} \arrow[dr, shift left, "0"] \arrow[dr, shift right, "1"'] & \\ {[1]} \arrow[u, shift left, "0"] \arrow[u, shift right, "1"'] \arrow[r, "\exists ! inv"'] & {[1]} \end{tikzcd}$$

$$\& \quad \begin{tikzcd} {[0]} \arrow[dr, shift left, "1"] \arrow[dr, shift right, "1"'] & \\ {[1]} \arrow[u, shift left, "0"] \arrow[u, shift right, "1"'] \arrow[r, "\exists ! i"'] & {[0]} \end{tikzcd}$$

inducing in a $\mathbb{T}$-model X, the str of a groupoid on it's underlying graph $X[1] \overset{x_0}{\underset{x_1}{\rightrightarrows}} X[0]$ with maps

$$X[1] \times_{X[0]} X[1] \cong X[2] \xrightarrow{X[c]} X[1],$$

$$X[0] \xrightarrow{X[i] \text{ - ids. map}} X[1],$$

$$X[1] \xrightarrow{X[inv]} X[1].$$

Uniqueness of the liftings involved in contractibility ensures associativity etc, so we really obtain a groupoid.

In fact we get a functor

$$\begin{tikzcd}
\mathrm{Mod}(\Pi) \arrow[rr, "K"] \arrow[dr] & {} \arrow[d, phantom, "="] & \mathrm{Mod}(\Pi_{Gpd}) \cong \mathrm{Gpd} \arrow[dl] \\
& {[\Delta_0^{op}, \mathrm{Set}]} &
\end{tikzcd}$$

commuting with the forgetful functors to $[\Delta_0^{op}, \mathrm{Set}]$,

which is induced by a commutative triangle

$$\begin{tikzcd}
& \Delta_0 \arrow[dl, "I"'] \arrow[dr, "I"] & \\
\Pi_{Gpd} \arrow[rr, "J"'] & & \Pi
\end{tikzcd}$$

(such is called a morph. of graphical theories). This commutative triangle is unique.

That is,

Theorem) $\mathbb{T}$Gpd is the initial contractible graphical theory.

Proof) - To say that $\mathbb{T}$ is contractible is equally to say that

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & {[1]} \end{tikzcd}$$

is a coproduct in $\mathbb{T}$.

- Since $[n]$ is a globular sum, it follows that

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2, "0"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "n"'] & {[n]} \end{tikzcd}$$

- picks out 0

- picks out n

is an $(n+1)$-fold coprod. in $\mathbb{T}$

- By commutativity in

$$\begin{tikzcd} & \Delta_0 \arrow[dl, "I"'] \arrow[dr, "I'"] & \\ \mathbb{F} \arrow[rr, "J"'] & & \mathbb{T} \end{tikzcd}$$

, J is forced to preserve these coproducts, so given a Function $n \xrightarrow{f} m \in \mathbb{F}$ we must define $Jf : Jn \longrightarrow Jm$ to be the unique map s.t.

$Jf \circ Ji = J(f \circ i)$ for $i \in \{0, \dots, n\}$.

Functoriality is straightforward. $\square$

Lecture 2 - Grothendieck ∞-groupoids

- Just as cats & groupoids are graphs w' structures, so ∞-cats / ∞-groupoids are ∞-graphs (aka globular sets) with structure.
- The globular cat $\mathbb{G}$ is the category gen by the graph

$$\{\ 0 \overset{\sigma_1}{\underset{\tau_1}{\rightrightarrows}} 1 \overset{\sigma_2}{\underset{\tau_2}{\rightrightarrows}} 2 \ \ldots\ n \overset{\sigma_{n+1}}{\underset{\tau_{n+1}}{\rightrightarrows}} n+1 \ \ldots\ \}$$

satisfying the coglobularity relations

$$\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n \quad \&$$
$$\sigma_{n+1} \circ \tau_n = \tau_{n+1} \circ \tau_n .$$

- It follows that there are just 2 maps $n \overset{\sigma_{n,m}}{\underset{\tau_{n,m}}{\rightrightarrows}} m$ for $m > n$ & I write $n \overset{\sigma}{\underset{\tau}{\rightrightarrows}} m$ when context is clear.
- The category $[\mathbb{G}^{op}, Set]$ is the category of globular sets -

a globular set $X : \mathbb{G}^{op} \longrightarrow Set$ consists of

$$X(n+1) \overset{s_{n+1}}{\underset{t_{n+1}}{\rightrightarrows}} X(n) \ \ldots\ldots\ X(2) \overset{s_2}{\underset{t_2}{\rightrightarrows}} X(1) \overset{s_1}{\underset{t_1}{\rightrightarrows}} X(0)$$

sat the globularity relations

$$s_n \circ s_{n+1} = s_n \circ t_{n+1} \quad \&$$
$$t_n \circ s_{n+1} = t_n \circ t_{n+1} .$$

- Here $X(n)$ is the set of <u>n-cells</u> of X.

- In a globular set, we have

objects (or 0-cells): $x, y, z \ldots \in X(0)$

1-cells : $x \xrightarrow{f} y$ ~ i.e. $s_1 f \xrightarrow{f} t_1 f$

2-cells :
$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & y \end{tikzcd}$$
i.e.
$$\begin{tikzcd} s_0 s_1 f \arrow[r, "s_1 f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \alpha"] & t_0 s_1 f \arrow[d, equal] \\ s_0 t_1 f \arrow[r, "t_1 f"'] & t_0 t_1 f \end{tikzcd}$$

3-cells
$$\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\alpha \Downarrow \overset{\varphi}{\Rrightarrow} \Downarrow \beta"] & y \end{tikzcd}$$
etc ...

- The Yoneda embedding
$$Y : \mathbb{G} \longrightarrow [\mathbb{G}^{op}, Set]$$ sends
$$n \longmapsto Yn,$$
the free n-cell (sometimes called n-globe).

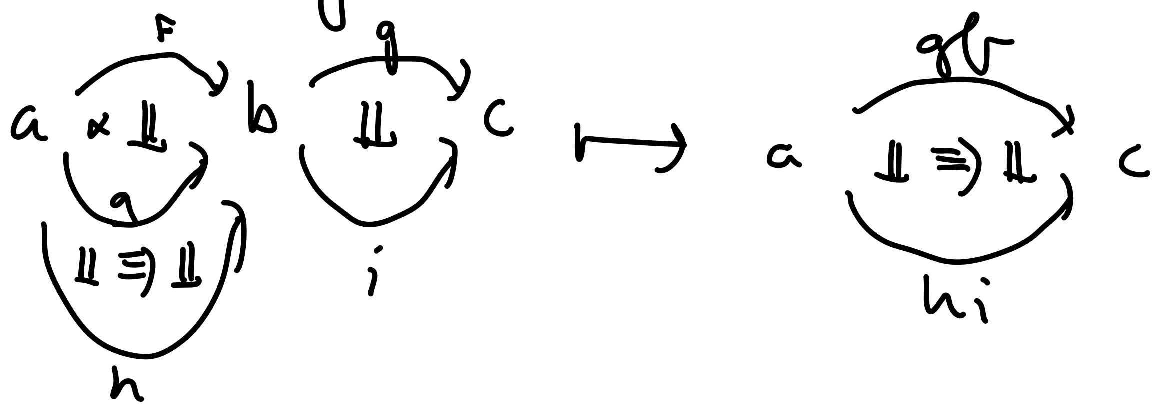
E.g. $Y(2) = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \right\}$, where I have omitted labels of cells (all are distinct)

- Given a diagram as below left in an ∞-cat / ∞-gpd

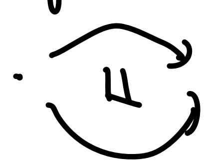
$$a \xrightarrow{f} b \xrightarrow{g} c \quad \longmapsto \quad a \xrightarrow{gf} c$$

we want to compose it.

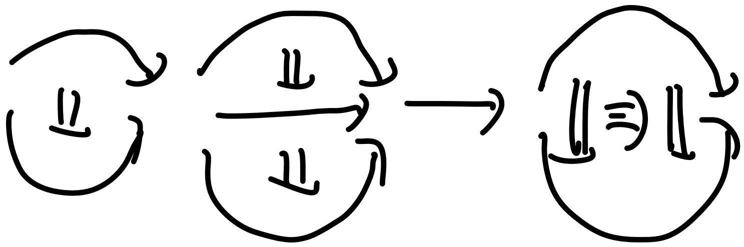
Similarly



- The diagram shapes (arities) are the so-called globular pasting diagrams (gpd's) & include globular sets like:

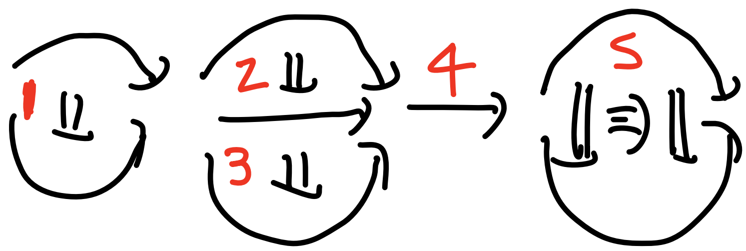
$\cdot \longrightarrow \cdot$,  . . . all n-cells

$\cdot \longrightarrow \cdot \longrightarrow \cdot$, $\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$, - - - -



How to parametrise such shapes?

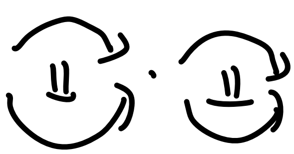
$(1,0,1,0,1) \sim \cdot \xrightarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{3} \cdot$

$(2,0,2,1,2,0,1,0,3) \sim$ 

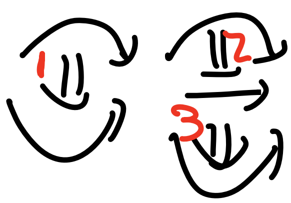
The sequence on the left is called a *table of dimensions* : it parametrises the associated globular pasting diag-

eg $(2,0,2,1,2,0,1,0,3)$ says -

attach a 2-cell 1 to a 2-cell 2 along a 0-cell

 ;

then attach a 2-cell 3 to 2 along its 1-cell boundary



& so on ...

Def) A table of dimensions is a sequence $\bar{n} = (n_1, \ldots, n_{2k+1})$ with

$$n_1 < n_2 > n_3 < n_4 > n_5 \cdots n_{2k-1} < n_{2k} > n_{2k+1}$$

Given a coglobular object $\mathbb{G} \xrightarrow{D} \mathcal{C}$, we obtain a diagram in $\mathcal{C}$

$$\begin{tikzcd}
& Dn_2 \arrow[dl, "D\sigma"'] \arrow[dr, "D\tau"] & & Dn_4 \arrow[dl, "D\sigma"'] \arrow[dr, "D\tau"] & & & Dn_{2k} \arrow[dl, "D\tau"'] \arrow[dr, "D\sigma"] & \\
Dn_1 & & Dn_3 & & \cdots & Dn_{2k-1} & & D_{2k+1}
\end{tikzcd}$$

whose colimit, if it exists, we call a globular sum $D\bar{n}$

$$
\begin{tikzcd}
 & Dn_2 \arrow[dl, "D\sigma"'] \arrow[dr, "D\tau"] & & Dn_4 \arrow[dl, "D\sigma"'] \arrow[dr, "D\tau"] & & & Dn_{2k} \arrow[dl, "D\tau"'] \arrow[dr, "D\sigma"] & \\
Dn_1 \arrow[drrr, "p_1"'] & & Dn_3 \arrow[dr, "p_3"] & & \cdots & Dn_{2k-1} \arrow[dll] & & D_{2k+1} \arrow[dllll, "p_{2k+1}"] \\
 & & & D\bar{n} & & & &
\end{tikzcd}
$$

- In particular, for $Y : \mathbb{G} \longrightarrow [\mathbb{G}^{op}, Set]$, $Y\bar{n}$ is the corresponding g.p.d.

E.g. $Y(1,0,2) =$
$$
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[out=60, in=120, loop] \arrow[out=-60, in=-120, loop] \arrow[phantom, "\Downarrow\Downarrow"]
\end{tikzcd}
$$

- We write Θ_0 for the cat whose objects are the tables of dimensions & with
$$\Theta_n(\bar{n}, \bar{m}) = [\mathbb{G}^{op}, Set](Y\bar{n}, Y\bar{m}).$$

- Equiv, Θ_0 is skeletal full subcat of $[\mathbb{G}^{op}, Set]$ containing the gpds.

- Θ_0 is our category of arities.

- In partic., have

$$\begin{array}{ccccc} n & \longmapsto & (n) & \longmapsto & Yn \\ \mathbb{G} & \xrightarrow{I} & \Theta_0 & \hookrightarrow & [\mathbb{G}^{op}, Set] \end{array}$$

fully faithful

Remark

① Given $D : \mathbb{G} \to \mathcal{C}$, globular sums $D(\bar{n})$ are equally the weighted colimits
$Y(\bar{n}) * D$.

② - A t.o.d. has dimension d where d is the maximum nat. no. appearing in the sequence.

E.g. $\dim(1,0,2) = 2$.

- There is just one t.o.d. of dimension 0,
$(0) = \bullet$

The 1-d t.o.ds are those of the form $(1,0,1,0,1,\ldots)$ ie.

$\cdot \to \cdot \to \cdot \to \cdots$ capturing the finite ordinals.

In particular $\Delta_0 \subseteq \Theta_0$ is the full subcategory containing the tods of dimension ≤ 1.

Def^{n}) A globular theory is an id. on obs functor $J : \Theta_0 \to \mathbb{T}$ preserving globular sums.

Given a globular theory $\mathbb{T}$, the category of $\mathbb{T}$-models in $\mathcal{C}$ is the full subcategory
$$\mathrm{Mod}(\mathbb{T}, \mathcal{C}) \hookrightarrow [\mathbb{T}^{op}, \mathcal{C}]$$
whose obs are those functors $\mathbb{T}^{op} \xrightarrow{X} \mathcal{C}$ sending globular sums to globular products - ie. sending the specified colimits to limits.

- We write $\mathrm{Mod}(\mathbb{T})$ for the category of $\mathbb{T}$-models in $\mathbf{Set}$.
- Of course, each representable $\mathbb{T}(-, \bar{n})$ is a $\mathbb{T}$-model, since reps send all colimits to limits.

- Restricting along

$$\begin{tikzcd} \mathbb{G} \arrow[r, "I"] \arrow[rr, bend right, "D"'] & \Theta_0 \arrow[r, "J"] & \mathbb{T} \end{tikzcd}$$

induces a forgetful functor

$$\mathrm{Mod}(\mathbb{T}) \xrightarrow{U} [\mathbb{G}^{op}, \mathrm{Set}]$$

so each $\mathbb{T}$-model has und. glob. set

- The category $\mathbb{G}$-Th of globular theories has morphisms given by commutative triangles

$$\begin{tikzcd} & \Theta_0 \arrow[dl, "J_{\mathbb{T}}"'] \arrow[dr, "J_{\mathbb{S}}"] \arrow[d, phantom, "="] & \\ \mathbb{T} \arrow[rr, "K"'] & {} & \mathbb{S} \end{tikzcd}$$

- Such automatically preserve globular sums.

Contractible globular sets

- In a globular set X, n-cells α, β are said to be parallel if
- $n = 0$ or $s_n \alpha = s_n \beta$ & $t_n \alpha = t_n \beta$.

picture
$$\begin{tikzcd} x \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & y \end{tikzcd}$$

A glob. set X is <u>contractible</u>

- if $X(0)$ is non-empty &
- given parallel $\alpha, \beta \in X(n)$
  $\exists\ \theta \in X(n+1)$ st $s_{n+1}(\theta) = \alpha$ & $t_{n+1}(\theta)$)

ie. given 0-cells x, y $\exists\ x \xrightarrow{f} y$.

Given 1-cells $\begin{tikzcd} x \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & y \end{tikzcd}$

$\exists$ $\begin{tikzcd} x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D] & y \end{tikzcd}$ etc.

Contractible Theories

Consider a glob theory $J : \Theta_0 \longrightarrow \mathbb{T}$,
with $D = J \circ I : \mathbb{G} \longrightarrow \mathbb{T}$
$n \longmapsto (n)$

Def^n) A globular theory $\mathbb{T}$ is contractible if each globular set
$U\mathbb{T}(-,\bar{m}) = \mathbb{T}(D-,\bar{m})$
is contractible.

What does this mean in elementary terms?

- Well $\Theta_0((0),\bar{m}) \subseteq \mathbb{T}((0),\bar{m})$ so $\mathbb{T}(D0,\bar{m})$ always non-empty.
- Let us call elements of $\mathbb{T}(Dn,\bar{m})$ n-cells in $\bar{m}$.
- Two such $Dn \overset{f}{\underset{g}{\rightrightarrows}} \bar{m}$ are parallel in $\mathbb{T}(D-,\bar{m})$
  $\iff$ $n=0$ or $f \circ D\sigma_n = g \circ D\sigma_n$ &
  $f \circ D\tau_n = g \circ D\tau_n$.

- Contractibility of Π says that given $\bar{m}$ a glob. sum &

$$\begin{tikzcd} D_n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "D\sigma"'] \arrow[d, shift left, "D\tau"] & \bar{m} \\ D(n+1) \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$

parallel in Π

Defn) A Grothendieck ∞-groupoid is a model for some contractible globular theory Π.

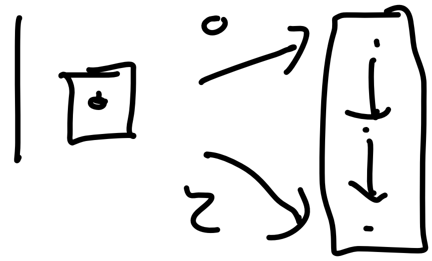
Remark) First outlined by Grothendieck in his letter to Daniel Quillen 1983 at start of Pursuing Stacks.
Exposed & made fully precise by Maltsiniotis in 2010.

So the idea is that models of such Π should have structure of a weak ∞-groupoid.

Let's investigate why?

Well in Θ_0, have maps

$$(0) \underset{2}{\overset{0}{\rightrightarrows}} (1,0,1)$$



so in Π

$$\begin{tikzcd}
(0) \arrow[r, shift left, "D\delta_0"] \arrow[r, shift right, "D\delta_2"'] \arrow[d, shift right, "D\sigma"'] \arrow[d, shift left, "D\tau"] & (1,0,1) \\
(1) \arrow[ur, dashed, "\exists c_0^1"'] &
\end{tikzcd}$$

which in a <u>Π-model X</u> gets sent to

$$\hat{G}(\cdot \to \cdot \to \cdot, X) \cong X(1,0,1) \xrightarrow{Xc_0^1} X1 \cong \hat{G}(\cdot \to \cdot, X)$$

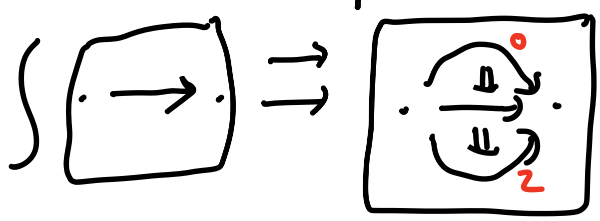
giving composition.

- Have

$$\begin{tikzcd}
(0) \arrow[r, shift left, "D\tau"] \arrow[r, shift right, "D\sigma"'] \arrow[d, shift right, "D\sigma"'] \arrow[d, shift left, "D\tau"] & (1) \\
(1) \arrow[ur, "\exists \mathrm{in}"'] &
\end{tikzcd}$$

corresponding to inverses map

$$(1) \underset{2}{\overset{0}{\rightrightarrows}} (2,1,2) \in \Theta_0$$



parallel, so

$$\begin{tikzcd}
(1) \arrow[r, shift left, "D\delta_0"] \arrow[r, shift right, "D\delta_2"'] \arrow[d, shift right, "D\sigma"'] \arrow[d, shift left, "D\tau"] & (2,1,2) \\
(2) \arrow[ur, dashed, "c_1^2"'] &
\end{tikzcd}$$

giving comp. of 2-cells along 1-cells.

Similarly get composition

$(n) \xrightarrow{c_i^n} (n, i, n)$ of n-cells along i-cell boundaries

all $i < n$.

- I call all these complexity 1 operations

- At level 2, we get whiskering ops

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & \bullet \arrow[r, "h"] & \bullet \end{tikzcd} \longmapsto \begin{tikzcd} \bullet \arrow[r, bend left, "hf"] \arrow[r, bend right, "hg"'] \arrow[r, phantom, "h\alpha\Downarrow"] & \bullet \end{tikzcd}$$

via

$$\begin{tikzcd} (1) \arrow[r, "c_0^1"] \arrow[d, shift right, "D_0"'] \arrow[d, shift left, "D_1"] & (1,0,1) \arrow[r, shift left, "\text{source 1-cell}"] \arrow[r, shift right, "\text{target 1-cells}"'] & (1,0,2) \\ (2) \arrow[urr, "\exists"] & & \end{tikzcd}$$

- For $\xrightarrow{f}\xrightarrow{g}\xrightarrow{h} \in X$,

we get the associator $(hg)f \Rrightarrow h(gf)$ via the lifting

$$
\begin{tikzcd}
(1) \arrow[r, "c_0^1"] \arrow[d, shift right, "\sigma"'] & (1,0,1) \arrow[r, shift left, "{(c_0^1,0,1)}"] \arrow[r, shift right, "{(1,0,c_0^1)}"'] & (1,0,1,0,1) \\
(2) \arrow[u, shift right, "\tau"'] \arrow[urr, "\exists"'] & &
\end{tikzcd}
$$

I call them <u>complexity 2 operations</u>
because they are dependent upon
complexity 1 operations

$\boxed{\text{Exercise}}$ Convince yourself that any
contractible globular theory
Π encodes the structure
you would like in a weak ω-groupoid.

Examples

- We can construct the globular theories of strict ω-categories Θ, by factoring

$$\begin{tikzcd} \Theta_0 \arrow[r, hook] \arrow[dr] & \hat{\mathbb{G}} \arrow[r, "F"] & \text{Strict } \infty\text{-cat} \\ & \Theta \arrow[ur] & \end{tikzcd}$$

called Joyal's cat Θ. Will re-appear later in defs of weak ∞-cat & (∞, n)-cat

- It admits a simple description, in fact, using "wreath products". (Later)

- The globular theory Θ_{gr} for strict ∞-groupoids does not admit a known simple description.

  Eg. - free ∞-groupoid on $(2) = \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}$ has infinitely many 1-cells, so

  $\Theta_{gr}((1),(2))$ is infinite ...

- Takes some work to show it is contractible (Ara - Strict ∞-groupoids are Grothendieck..)

Lecture 3

Last time I mentioned

Proposition

Consider $D : \mathbb{G} \longrightarrow \mathcal{C}$ where $\mathcal{C}$ has D-glob. sums.

Then

$$\begin{tikzcd} \Theta_0 \arrow[dr, "\mathrm{lan}_I D"] & \\ \mathbb{G} \arrow[u, "I"] \arrow[r, "D"'] \arrow[ur, phantom, "\cong"] & \mathcal{C} \end{tikzcd}$$

exists and sends $\bar{n} \longmapsto D(\bar{n})$.

It preserves I-globular sums, & is the unique up to iso extension w/ this property.

$\boxed{\text{Proof}}$ The following proof is probably too formal!

- From enriched cat. theory, the pwise left Kan ext. will exist $\Longleftrightarrow$ the weighted colims $\Theta_0(I-,\bar{m}) * D$ exist $\forall\, \bar{m}$, where $\Theta_0(I-,\bar{m}) : \mathbb{G}^{op} \to \mathrm{Set}$.
- It is then defined as the ess. unique functor sending the canonical cocone
  $$\Theta_0(I-,\bar{m}) \xrightarrow{1} \Theta_0(I-,\bar{m})$$
  to a weighted colimit in $\mathcal{C}$.
  But $\Theta_0(I-,\bar{m}) \cong$ (applying $J : \Theta_0 \to [\mathbb{G}^{op},\mathrm{Set}]$)
  $[\mathbb{G}^{op},\mathrm{Set}](Y-, Y(\bar{m})) \cong$ (by Yoneda)
  $Y(\bar{m})$ & then
  $$Y(\bar{m}) \cong \Theta_0(I-,\bar{m}) \text{ is the cocone exhib.}$$
  $$\bar{m} \text{ as } Y(\bar{m}) * I.$$
- But since weighted cols of the $Y(\bar{m})$ are precisely the globular sums, the result follows. $\square$

The globular theory of topological spaces

- Firstly, we will construct the globular theory of spaces & show it is contractible.
- Firstly, we construct

$$\begin{array}{rcl} D : \mathbb{G} & \longrightarrow & Top \\ n & \longmapsto & D^n = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \} \end{array}$$

$$\rotatebox{90}{$=$}$$

$\bullet$, -1 —— 1 , 

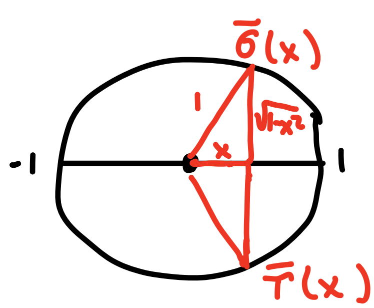
Then

$$n \underset{\tau}{\overset{\sigma}{\rightrightarrows}} n+1 \longmapsto D^n \underset{\bar{\tau}}{\overset{\bar{\sigma}}{\rightrightarrows}} D^{n+1}$$

are the north/south hemisphere maps

$$\bar{\sigma}(x) = (x, \sqrt{1-x^2}), \quad \bar{\tau}(x) = (x, -\sqrt{1-x^2})$$

eg.



- Then we obtain the Kan extension

$$\begin{aligned} D : \Theta_0 &\longrightarrow \mathrm{Top} \text{ sending} \\ \bar{n} &\longmapsto D(\bar{n}). \end{aligned}$$

We obtain the globular theory Π_{Top} of spaces by factoring

$$\begin{tikzcd} \Theta_0 \arrow[r, "I"] \arrow[rr, bend right, "D"'] & \Pi_{\mathrm{Top}} \arrow[r, "J"] & \mathrm{Top} \end{tikzcd}$$

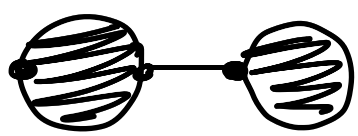
as id. on obs / FF.

Proposition

Π_{Top} is contractible.

Proof

- The spaces $D(\bar{n})$ are things like

 etc.

$D(\bar{n})$ is always contractible as a space, ie $D(\bar{n}) \to 1$ a homotopy equivalence.

We must show that given

$$
\begin{tikzcd}
D^n \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[d, shift right, "\bar{\sigma}"'] \arrow[d, shift left, "\bar{\tau}"] & & D(\bar{m}) \\
D^{n+1} \arrow[urr, "\exists h"'] & &
\end{tikzcd}
$$

parallel

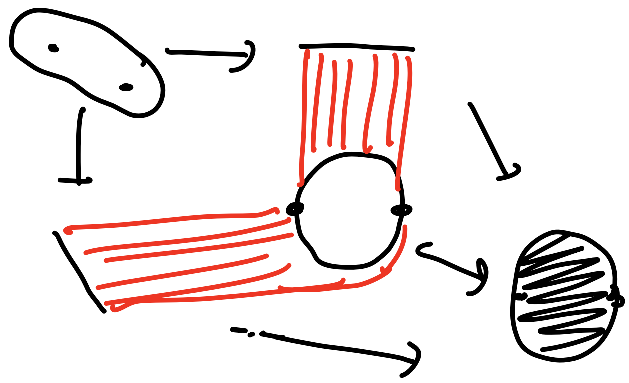
Now to say that f & g are parallel is, by definition, to say that

$$
\begin{tikzcd}
D^{n-1} + D^{n-1} \arrow[r, "\bar{\sigma} + \bar{\tau}"] \arrow[d, "\bar{\sigma} + \bar{\tau}"'] & D^n \arrow[d, "f"] \arrow[dl, phantom, "="] \\
D^n \arrow[r, "g"] & D(\bar{m})
\end{tikzcd}
$$

Note that $S^n = \{ x : |x| = 1 \}$ is the pushout

$$\begin{tikzcd}
D^{n-1}+D^{n-1} \arrow[r, "\bar\sigma+\bar\tau"] \arrow[d, "\bar\sigma+\bar\tau"'] & D^n \arrow[d, hook, "i"] \arrow[ddr, "\bar\sigma"] & \\
D^n \arrow[r, hook, "j"'] \arrow[drr, "\bar\tau"'] & S^n \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook, "k"] & \\
 & & D^{n+1}
\end{tikzcd}$$

eg.



So using pushout property

obtain

$$\begin{tikzcd}
D^n \arrow[d, shift right, "i"'] \arrow[d, shift left, "j"] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & \\
S^n \arrow[r, "\langle f, g \rangle"] \arrow[d, hook, "k"'] & D(\bar n) \arrow[d, "!"] \\
D^{n+1} \arrow[r, "!"'] & 1
\end{tikzcd}$$

But now k is a cofibration in the Quillen model structure on Top, & $D(\bar{n}) \to 1$ a triv. fibration

So

$$\begin{tikzcd}
D^n \arrow[d, shift right, "i"'] \arrow[d, shift left, "j"] \arrow[drr, bend left, "f"] \arrow[drr, "g"'] & & \\
S^n \arrow[rr, "{\langle f, g \rangle}"] \arrow[d, hook, "k"', "\text{cof}"'] & & D(\bar{n}) \arrow[d, "!", "\text{triv fib}"'] \\
D^{n+1} \arrow[rr, "!"'] \arrow[urr, dashed, "\exists"] & & 1
\end{tikzcd}$$

proving contractibility. $\square$

Fundamental ∞-groupoid of a space

Now $\Theta_0 \xrightarrow{I} \Pi_{Top} \xrightarrow{J} Top$ induces

$$Top \xrightarrow{N_J} [\Pi_{Top}^{op}, Set]$$

& $N_J X = Top(J-, X) = Top(-, X) \circ J$

sends glob. sums to globular products

since J pres. glob. sums, $Top(-,X)$ sends colims to lims.

Hence each $N_J X$ is a model of Π_{Top} — we obtain a factorisation

$$Top \xrightarrow{N_J} Mod(\Pi_{Top}) \hookrightarrow [\Pi_{Top}^{op}, Set]$$

$N_J X$ is the fundamental ∞-groupoid of X.

- What does it look like?

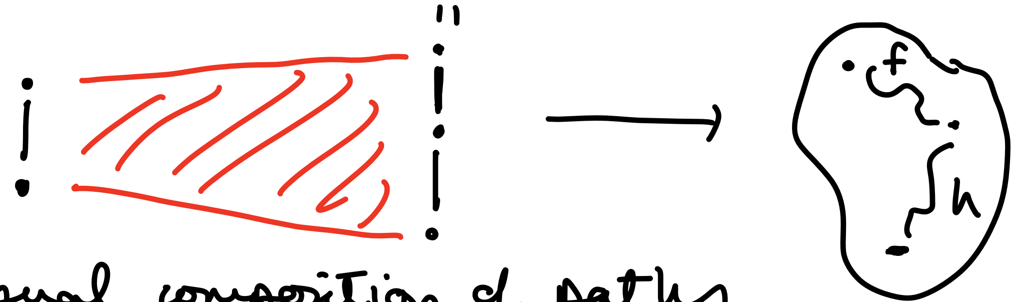
$N_J X(\bar{n}) = Top(D(\bar{n}), X)$.

- In partic, underlying globular set has

$$N_J X(n) = \mathrm{Top}(D^n, X).$$

- <u>composition</u>

$$D(1) \xrightarrow{c_0^1} D(1,0,1) \xrightarrow{\langle f, g \rangle} X$$



is usual composition of paths (up to htpy).

So it really does look like the <u>fundamental ∞-groupoid</u> should look.

Weak equivalences of ∞-groupoids

What is a weak equiv. of top spaces?
Usually defined as
$f : X \to Y$ inducing
- $\pi_0 X \longrightarrow \pi_0 Y$ bij^n on path comps
- bij $\pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ on homotopy groups $\forall n, x \in X$.

Remark: group structure not actually relevant to this def...

- Similarly, can define homotopy groups of ∞-groupoids.
- If X is model of contractible theory Π, let $\pi_0 X$ = set of connected components of underlying graph of X.

Also, for all n can form groupoid $n\text{-}Gr(X)$ whose
- objects are n-cells α, β
- morphisms: equiv. classes of $(n+1)$-cells

e-rel $\alpha \xrightarrow{\varphi} \beta$ where .

where $\varphi \sim \varphi'$ if $\exists\,(n+2)$-cell $\varphi \to \varphi'$.

Exercise

- Each $n\text{-}Gr(X)$ is a groupoid, using the composition operations in Π

eg $\quad (n) \xrightarrow{c^n_{n-1}} (n, n-1, n) \in \Pi$

& this is independent of the choice of composition operation.

- Now given $x \in X_0$, obtain $k_n(x) \in X_n$, where $k_n(x)$ is the "identity n-cell" on x, arising from the contractibility of Π

Then $\Pi_{n+1}(X, x) := n\text{-}Gr(X)(k_n x, k_n x)$

Eg. $\pi_1(X,x)$ has

$$\begin{tikzcd} x \arrow[r, bend left, "k_1 x"] \arrow[r, bend right, "k_1 x"'] \arrow[r, phantom, "{[\alpha]} \Downarrow"] & x \end{tikzcd}$$

e-class

This gives a functor

$\pi_0 : \mathrm{Mod}(\Pi) \longrightarrow \mathrm{Set}$ &

$\pi_n : \mathrm{Mod}(\Pi)_* \longrightarrow \mathrm{Grp}$

pointed models
& point preserving maps

taking the homotopy groups of a Π-model.

- In fact, we have

$$\begin{tikzcd} (\mathrm{Top},*) \arrow[rr, "N_J"] \arrow[dr, "\pi_n"'] & \arrow[d, phantom, "="] & (\mathrm{Mod}(\Pi_{\mathrm{Top}}),*) \arrow[dl, "\pi_n"] \\ & \mathrm{Grp} & \end{tikzcd}$$

Def^n) A morphism $f : X \to Y$ of Π-models is a weak equiv.

if
- $\Pi_0 X \longrightarrow \Pi_0 Y$ bij^n on path comps
- bij $\Pi_n(X, x) \longrightarrow \Pi_n(Y, f(x))$ $\forall n, x \in X_0$.

Corollary

$$\mathrm{Top} \xrightarrow{N_J} \mathrm{Mod}(\Pi_{\mathrm{Top}})$$

preserves & reflects weak equivalences.

Proof

Since the triangle commutes.

The homotopy hypothesis?

- A good guess for what the homotopy hypothesis should say is that the above functor induces an equiv. of cats when we invert the weak equivs on either side.
- Close, but not quite Grothendieck's formulation anyway ...

Weakness (aka cellularity)

So far, we talked about contractibility which leads to ∞-groupoids.

But suppose we want to single out the theories of weak ∞-groupoids?

How to do it?

Grothendieck's formulation was (essentially):

Π is a coherator if it is contractible & Π is the colimit of a chain

$$\Theta_0 = \Pi_0 \to \Pi_1 \to \dots \to \Pi_n \xrightarrow{J_n} \Pi_{n+1} \to \dots \to \Pi \in \mathcal{G}\text{-Th}$$

where · there is a set P_n of parallel pairs $(\ell) \overset{u}{\underset{v}{\rightrightarrows}} \overline{m}$ in Π_n

such that Π_{n+1} is obtained by freely adding a lifting

$$\begin{tikzcd} (\ell) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, shift left] \arrow[d, shift right] & \overline{m} \\ (\ell+1) \arrow[ur, "\varphi_{u,v}"'] & \end{tikzcd}$$

$\varphi_{u,v} \in \Pi_{n+1}$

for each $(u,v) \in P_n$.

What is the idea?

E.g. if we take all parallel pairs at each stage, then:

in Π_1, get liftings such as

$$\begin{tikzcd}
(0) \arrow[d, shift right] \arrow[d, shift left] \arrow[drr, shift left, "0"] \arrow[drr, shift right, "2"'] & & \\
(1) \arrow[rr, "0_0^1 \in \Pi_1"'] & & (1,0,1)
\end{tikzcd}$$

& then

$$(1) \xrightarrow{0_0^1} (1,0,1) \underset{(1,\,0_0^1)}{\overset{(0_0^1,\,1)}{\rightrightarrows}} (1,0,1,0,1)$$

is a parallel pair in Π_1 but not equal since we added the liftings freely.

Thus we don't force any equations like associativity.

More conceptually, it corresponds to cellularity, which is just another part of the same story as contractibility.

Cellularity & contractibility

- let J be a class of morphisms in a cat $\mathcal{C}$.

Write $j \perp f$ if
$$\begin{tikzcd} a \arrow[r, "r"] \arrow[d, "j"'] & c \arrow[d, "f"] \\ b \arrow[r, "s"'] \arrow[ur, "\exists"] & d \end{tikzcd}$$

& $J^{\square} = \{ f : j \perp f \ \forall j \in J \}$
& ${}^{\square}J = \{ f : f \perp j \ \forall j \in J \}$

Then $J \subseteq {}^{\square}(J^{\square})$.

- We say $({}^{\square}(J^{\square}), J^{\square})$ is a weak factorisation system if each arrow factors as a

J-cofibration (in ${}^{\square}(J^{\square})$)
followed by a
J-contraction (in $J^{\square}$)

- These can be generated using Quillen's small object argument.
if. eg J is a set & $\mathcal{C}$ is nice,
eg. locally presentable,

This produces factorisations

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "g"'] & & b \\ & c \arrow[ur, "h"'] & \end{tikzcd}$$

$${}^{\square}(J^{\square}) \supseteq J\text{-cell} \ni g \qquad h \in J^{\square}$$

J-cell is the closure of $J \subseteq \mathrm{Mor}(\mathcal{C})$
under - pushouts of maps in J
- coproducts
- transfinite composition

- Then the J-cofibrations are the retracts of the J-cellular maps.

- We say an object X is
  - J-cellular if $\emptyset \to X$ is a J-cellular map
  - J-contractible (or J-injective) if

$$X \to 1 \in J^{\square}.$$

i.e.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "j"'] & X \\ b \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad J \ni j \perp$$

Next time, I will describe a set of maps $\mathcal{B}$ in the category $\mathbb{G}$-Th such that:

- the (J-cellular, J-contractible)-objects are exactly Grothendieck's coherators, i.e. the globular theories for weak ∞-groupoids.

Lecture 4 - Cellularity & the small object argument

- Consider $J \subseteq \mathrm{Mor}(\mathcal{C})$.
- The small object argument will factor

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & C \arrow[ur, "h"'] & \end{tikzcd}$$

$${}^{\square}(J^{\square}) \supseteq \mathrm{Cell}(J) \ni g, \qquad h \in J^{\square}$$

- I just want to explain this in the case $B = 1$: then for each $A \in \mathcal{C}$ we form

$$A \xrightarrow{f \,\in\, \mathrm{Cell}(J)} A^* \in \mathrm{Inj}(J) .$$

- This is weakly universal in the sense that given

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & B \in \mathrm{Inj}(J) \\ & A^* \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Indeed

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "{}^{\square}(J^{\square}) \ni f"'] & B \arrow[d, "! \in J^{\square}"] \\ A^* \arrow[r, "!"'] \arrow[ur, dashed, "\exists"] & 1 \end{tikzcd}$$

but we will also explain how to

make it <u>really universal</u>.

- I will also assume that each $j: A \longrightarrow B \in J$ has A a <u>finitely presentable object</u>, which implies each map f to a colim of a chain

$$\begin{tikzcd}[column sep=small]
 & & & A \arrow[d, "\exists"'] \arrow[drrrr, "f"] & & & & \\
B_0 \arrow[r] & B_1 \arrow[r] & \cdots \arrow[r] & B_n \arrow[r] & B_{n+1} \arrow[r] & \cdots \arrow[r] & B_\omega
\end{tikzcd}$$

factors through some earlier stage.

- Also that $\mathcal{C}$ is <u>locally small</u> & <u>cocomplete</u>.

<u>The classical small object argument</u>

(detailed explanation)

- Consider $X \in \mathcal{C}$.
- Need to find $X \longrightarrow X^*$ w/ X^+ J-inj.

- Consider the solid part of

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X \arrow[d, "\eta_X"] \\ B \arrow[r, dotted] & X^* \end{tikzcd}.$$

Certainly we need a dotted filler, so might define X^* as universal' ob. equipped with arrow $X \xrightarrow{\eta_X} X^*$ & filling function φ as below

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X \arrow[d, "\eta_X"] \\ B \arrow[r, "\varphi(j,f)"] & X^* \end{tikzcd}.$$

Its universal property is that given a second pair $(X \xrightarrow{k} Y, \Theta)$

filling function

$\exists!\ X^* \xrightarrow{k'} Y$ such that

$$
\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d] & X \arrow[d, "\eta_X"] \arrow[ddr, bend left, "k"] & \\
J \ni j & B \arrow[r, "\varphi(j,f)"] \arrow[drr, bend right, "\Theta(j,f)"'] & X^* \arrow[dr, "k'"] \arrow[d, phantom, "="] & \\
 & & \, & Y
\end{tikzcd}
$$

This can be captured as the pushout on right below

$$
\begin{tikzcd}
A \arrow[r, "u(j,f)"] \arrow[d, "j"'] \arrow[rrr, bend left, "f"] & \sum\limits_{\substack{(j:A\to B\in J,\\ f:A\to X)}} A \arrow[rr, "\varepsilon"] \arrow[d, "\sum j"] & & X \arrow[d, "\eta_X"] \\
B \arrow[r, "v(j,f)"] \arrow[rrr, bend right, "\theta(j,f)"'] & \sum\limits_{\substack{(j:A\to B\in J,\\ f:A\to X)}} B \arrow[rr] \arrow[urr, phantom, "\ulcorner" near end] & & X^*
\end{tikzcd}
$$

$\wr$ coprod inclusions $\wr$

so η_X is J-cellular.

Problem : given

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & X^* \\
B &
\end{tikzcd}
$$

if

f factors as $A \xrightarrow{f'} X \xrightarrow{\eta_X} X^*$,

get filler

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "j"'] \arrow[d, phantom, "\perp"] \arrow[dr, "f''"] & X \arrow[d, "\eta_X"] \arrow[d, phantom, "\perp"] \\ B \arrow[r, "\theta(j, f')"'] \arrow[ur, phantom, "="] & X^* \end{tikzcd}$$

but if $f : A \longrightarrow X^*$ does not factor through η_X, perhaps no filler - X^* not J-injective

So we repeat :

setting $X_0 = X$;

$X_{n+1} = (X_n)^*$

$$\begin{tikzcd} X = X_0 \arrow[r, "\eta_{X_0}"] \arrow[rrrr, bend right, "\eta_{0,\omega}"'] & X_1 \arrow[r, "\eta_{X_1}"] & X_2 \cdots \arrow[r] & X_n \cdots \arrow[r] & X_\omega \end{tikzcd}$$

& X_ω the colimit of the chain.

- Then x is J-cellular by construction.

- Consider

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X_\omega \\ B \end{tikzcd}$$

Then

$$\begin{tikzcd} & X_n \arrow[dr, "x_{n,\omega}"] & \\ A \arrow[ur, "\exists f'"] \arrow[rr, "f"] & & X_\omega \end{tikzcd}$$

as A finitely pres.

so

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "j"'] \arrow[drr, bend left=60, "f"] \arrow[dr, phantom, "//"] & X_n \arrow[d, "x_{n,n+1}"] \arrow[dr, "x_{n,\omega}"] & \\ B \arrow[r, "\varphi^*(j,f')"'] & X_{n+1} \arrow[r, "x_{n+1,\omega}"'] \arrow[ur, phantom, "//"] & X_\omega \end{tikzcd}$$

gives filler.

Thus $X_\omega \in \mathrm{Inj}(J)$ & this completes the usual small object argument.

- The small object argument has some odd features.
- At stage 1, we add a canonical lifting

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[r, "n_{0,1}"] & X_1 \\ B \arrow[urr, "\varphi^1(j,f)"'] & & \end{tikzcd} \qquad J \ni j$$

& then at stage 2 a lifting

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[r, "n_{0,1}"] & X_1 \arrow[r, "n_{1,2}"] & X_2 \\ B \arrow[urrr, "\varphi^2(j, n_{0,1} \circ f)"'] & & & \end{tikzcd} \qquad J \ni j$$

So now two liftings

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X \arrow[r, "\eta_{0,1}"] & X_1 \arrow[r, "\eta_{1,2}"] & X_2 \\ B \arrow[urr, "\varphi^1(j,f)"] \arrow[urrr, "\varphi^2(j,\eta_{0,1}\circ f)"'] & & & \end{tikzcd}$$

?

For the same problem which need not be the same !!

In particular this means that in X_ω we have added many fillers for the same lifting problem - this prevents X_ω from having canonical liftings / a universal property.

There are two solutions

① This involves forming coequalisers

identifying the liftings $\to$ the algebraic small object argument.

(2) A simpler solution is what I'll call the efficient small object argument:

it is simpler than (1), but in the cases we are interested in, they coincide.

The efficient small object argument

This starts exactly as before:

$X_0 = X$,

$X_0 \xrightarrow{\eta_{0,1}} X_1$ is $X \xrightarrow{\eta_X} X^*$.

Suppose we have $X_n \xrightarrow{\eta_{n,n+1}} X_{n+1}$.

Call $A \xrightarrow{f} X_{n+1}$ irredundant if it does not factor through $X_n \xrightarrow{\pi_{n,n+1}} X_{n+1}$.

- We define X_{n+2} as the universal object equipped with a filler

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X_{n+1} \arrow[d, "\pi_{n+1,n+2}"] \\ B \arrow[r, "\varphi(j,f)"] & X_{n+2} \end{tikzcd}$$

for each pair (j,f) with f irredundant.

- Then $X_{n+1} \longrightarrow X_{n+2}$ is again a pushout of a coproduct of maps in J just as before, only we only consider irredundant f.

- Now take the colim of the chain

$$X \to X_1 \to \cdots \to X_n \longrightarrow X_e$$

as before -

an easy adaptation of the prev. proof shows X_e is J-injective

& $X \to X_e$ is J-cellular by construction.

In fact, under further assumptions X_e is the free algebraic injective.

Algebraic injectivity

A J-algebraic injective (X, φ) is an object $x \in \mathcal{C}$ + a lifting function

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j \in J"'] & X \\ B \arrow[ur, "\varphi(j,f)"'] \arrow[u, phantom, ""] \end{tikzcd}$$

A morphism $g : (X, \varphi) \to (Y, \theta)$ of algebraic injectives is $g : X \to Y$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j \in J"'] & X \arrow[r, "g"] & Y \\ B \arrow[ur, "\varphi(j,f)"'] \arrow[urr, bend right, "\theta(j, gf)"'] \end{tikzcd}$$

These form a cat **J-Alg**, which comes with a **forgetful functor**

$$U : J\text{-Alg} \longrightarrow \mathcal{C}.$$

Example

In Set, consider

$$j : 2 \hookrightarrow 3 \quad \& \quad J = \{j\}.$$

$$\boxed{\begin{matrix}0\\1\end{matrix}} \rightrightarrows \boxed{\begin{matrix}0\\1\\2\end{matrix}}$$

A J-alg. injective (X, φ) gives

$$\begin{tikzcd}
2 \arrow[r, "{(a,b)}"] \arrow[d, hook, "j"'] & X \\
3 \arrow[ur, "{(a,b,m(a,b))}"'] &
\end{tikzcd}$$

ie. a function $X^2 \xrightarrow{m} X$.

Thus J-Alg is the category of **magmas**.

More generally, any category

Ω-Alg for Ω a signature in universal algebra is of form

J-Alg for J a set of monos between finite sets

& let us compare the efficient & classical small object arguments.

Let $X \in \mathrm{Set}$. In both cases, X_1 is universally equipped with fillers

$$\begin{tikzcd} 2 \arrow[r, "{(a,b)}"] \arrow[d, hook] & X \arrow[d, hook] \\ 3 \arrow[r, "{(a,b,m(a,b))}"'] & X_1 \end{tikzcd}$$

so $X_1 = X \cup \{m(a,b) : a,b \in X\}$

- At stage 2, $2 \xrightarrow{(u,v)} X_1$ is <u>irredundant</u> just when at least one of u, v does not belong to X.
i.e. one is of form $m(a,b)$.

- So in efficient soa, we have fillers like

$$
\begin{tikzcd}
2 \arrow[r, "{(a,m(b,c))}"] \arrow[d, hook] & X_1 \arrow[d, hook] \\
3 \arrow[r, "{(a,m(b,c),m(a,m(b,c)))}"'] & X_2
\end{tikzcd}
$$

X_0 - a, b, c

X_1 - $a, b, m(a,b), \ldots$

X_2 - $m(a,m(b,c))$, $m(m(a,b),m(c,d))\ldots$

$X_\ell = \bigcup X_n$ is <u>free magma</u> on X!

- Classical soa produces $m(a,b)$, but also $m'(a,b)$ at stage 2 - useless..

This suggest efficient soa produces free algebraic injectives &, under some assumptions, it does.

Remark: One possible advantage of classical soa is due to its simplicity - simply iterating a functor $X \mapsto X^*$.

I have not checked functoriality of the efficient soa. Of course, under the assumptions below, it will be functorial - even a monad.

$\boxed{\text{Theorem}}$ Let J be a set of monos with f.p. domain & $\mathcal{C}$ cocomplete. Suppose J-cellular maps are mono.

Then $X \xrightarrow{n_{0,\omega}} X_\omega$ is the free J-algebraic injective on X.

Remark) In **Set** or $[\mathcal{C}, \mathbf{Set}]$ this holds. Main point is that pushouts of mono are mono. In Set, each mono of form $X \xrightarrow{i} X+Y$ &

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"', "\text{mono}"'] & Z \arrow[d, "\text{mono}"] \\ X+Y \arrow[r] & Z+Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

~~Proof~~ First, we give X_ω structure of object of J-Alg.

Given $f : A \longrightarrow X_\omega$, let's define the complexity of f as the least natural number st

f factors as
$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[dr, "f"'] & X_n \arrow[d, "\eta_{n,\omega}"] \\ & X_e \end{tikzcd}.$$

(Such an n exists as A fin. pres.)

By assumption, $\eta_{n,\omega}$ is mono – hence the factorisation f' is unique.

Given
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X_e \\ B \end{tikzcd}$$
where f has complexity n, we define

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "j"'] \arrow[rrd, bend left=40, "f"] \arrow[dr, phantom, "="] & X_n \arrow[d, "\eta_{n,n+1}"] \arrow[dr, "\eta_{n,\omega}"] \arrow[dr, phantom, "=", near start] & \\ B \arrow[r, "\varphi^{n+1}(j,f')"'] \arrow[rr, bend right=40, "\varphi(j,f)"'] & X_{n+1} \arrow[r, "\eta_{n+1,\omega}"'] \arrow[d, phantom, ":="] & X_e \end{tikzcd}.$$

Since n & f' are uniquely determined by f, $\varphi(j,f)$ is <u>well defined</u>.

Consider $(Y,\theta) \in J\text{-Alg}$ & $g : X \to Y \in \mathcal{C}$.

We must show $\exists!\ (X_e, \varphi) \xrightarrow{\bar{g}} (Y,\theta)$ such that

$$\begin{tikzcd} & X_e \arrow[dr, "\bar{g}"] & \\ X \arrow[ur, "n_{0,\omega}"] \arrow[rr, "g"'] \arrow[rr, phantom, shift left=3, "="] & & Y \end{tikzcd}$$

To give such an extension is to give a system

$$\begin{tikzcd} X_0 \arrow[d, "n_{0,1}"'] \arrow[drr, "g_0"] & & \\ X_1 \arrow[d] \arrow[rr, "g_1"] & & Y \\ \vdots & & \\ X_n \arrow[d, "n_{n,n+1}"'] \arrow[uurr, "g_n"] & & \\ X_{n+1} \arrow[uuurr, "g_{n+1}"'] & & \\ \vdots & & \end{tikzcd}$$

with $g_0 = g$

& to say that g preserves fillers of complexity n is to say that

$\forall\, A \xrightarrow{f} X_n$ non-redundant

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "="] & X_n \arrow[d, "\eta_{n,n+1}"] \arrow[dr, "\eta_{n,\omega}"] & & \\
B \arrow[r, "\varphi^{n+1}(j,f)"'] \arrow[rrr, bend right, "\theta(j,\, \bar{g}\cdot\eta_{n,\omega}\cdot f)"'] & X_{n+1} \arrow[r, "\eta_{n+1,\omega}"'] \arrow[u, phantom, ""] & X_e \arrow[r, "\bar{g}"] & Y
\end{tikzcd}$$

ie

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "="] & X_n \arrow[d, "\eta_{n,n+1}"] \arrow[dr, "g_n"] & \\
B \arrow[r, "\varphi^{n+1}(j,f)"'] \arrow[rr, bend right, "\theta(j,\, g_n f)"'] & X_{n+1} \arrow[r, "g_{n+1}"] & Y
\end{tikzcd}$$

- But by the universal property of X_{n+1}, given g_n $\exists !$ g_{n+1} extending g_n & having this property.
- Since $g_0 = g$, we hence obtain a unique extension $\bar{g}$ preserving fillers for morphisms of complexity n $\forall n$ – that is, preserving all fillers. $\square$

Closest thing to a ref for this stuff:

- JB - Iterated algebraic injectivity & the Faithfulness conjecture.

Builds on

- Nikolaus - Algebraic models for higher cats.

- For more on algebraic small obj. arg,

Garner - Understanding the small obj. argument

L5 - Cellularity continued & the homotopy hypothesis

Recap: J a set of arrows in $\mathcal{C}$

Then $J \subseteq \mathrm{Cell}(J) \subseteq \mathrm{Mor}(\mathcal{C})$ consists of closure of J in $\mathrm{Mor}(\mathcal{C})$ under coproducts, pushouts & transfinite composites.

Exercise: $\mathrm{Cell}(J)$ equally consists of the transfinite composites of pushouts of coproducts of maps in J.

(Idea: Just need to show this second class of maps is closed under coproducts, pushouts & transfinite composites.)

Last week: assuming $\mathcal{C}$ locally small, cocomplete, domains of maps in J finitely presentable the (efficient) small object argument produces factorisation

$$\begin{tikzcd} & A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ \mathrm{Cell}_\omega(J) : \exists & & C \arrow[ur, "h"] & \in J^{\square} \end{tikzcd}$$

ω-cellular maps:

the ω-composites of pushouts of coproducts of maps in J.

In fact, under above assumptions, $\mathrm{Cell}_\omega(J) = \mathrm{Cell}(J)$.

Proof of this a bit harder –
see Maltsiniotis "Grothendieck ω-groupoids"
Proposition A.6

Recall from L3:

$\mathbb{T}$ is a coherator if it is contractible & $\mathbb{T}$ is the colimit of a chain

$$\Theta_0 = \mathbb{T}_0 \to \mathbb{T}_1 \to \cdots \to \mathbb{T}_n \xrightarrow{J_n} \mathbb{T}_{n+1} \to \cdots \to \mathbb{T} \in G\text{-}Th$$

where - there is a set P_n of parallel pairs $(\ell) \overset{u}{\underset{v}{\rightrightarrows}} \overline{m}$ in $\mathbb{T}_n$ such that $\mathbb{T}_{n+1}$ is obtained by freely adding a lifting

$$\begin{tikzcd} (\ell) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, shift left] \arrow[d, shift right] & \overline{m} \\ (\ell+1) \arrow[ur, "\varphi_{u,v} \in \mathbb{T}_{n+1}"'] & \end{tikzcd}$$

for each $(u,v) \in P_n$.

Now I want to describe a set of maps J in the category of globular theories $G\text{-}Th$ such that

coherators $\equiv$ J-cellular, J-contractible theories

or ω-cellular

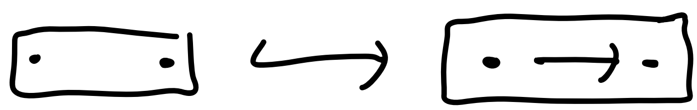
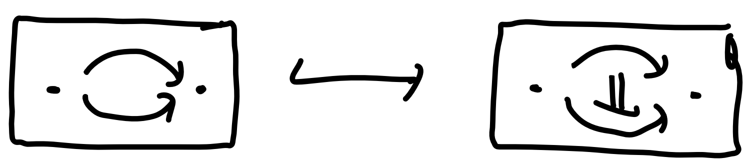
i.e. $\Theta_0 \longrightarrow \mathbb{T} \longrightarrow 1$

initial ob cellular contraction

Firstly, consider the category $[\mathbb{G}^{op}, \mathrm{Set}]$ & let $B = \{ \delta Y_n \xhookrightarrow{j^n} Y_n : n \in \mathbb{N} \}$

boundary of n-cell n-cell

defined by $\delta Y_n(m) = \emptyset$ if $m \geq n$
& $\delta Y_n(m) = Y_n(m)$ if $m < n$.

E.g.	$\delta Y_n \hookrightarrow Y_n$
$n = 0$	$\emptyset \longrightarrow 1$
$n = 1$	
$n = 2$	
	...

Then δY_n consists of a parallel pair of $(n-1)$-cells, in particular

X is $\mathcal{B}$-injective $\iff$ X is contractible

$$\begin{tikzcd} \delta Y_n \arrow[r, "\substack{\text{parallel } (n-1)\text{-cells} \\ x,y}"] \arrow[d] & X \\ Y_n \arrow[ur, dashed, "\exists\, x \to y \in X_n"'] & \end{tikzcd}$$

Remarks

① Each δY_n, Y_n is finite & so certainly <u>finitely presentable</u>.

② Each globular set X is $\mathcal{B}$-cellular: it is a colimit

$$\emptyset \longrightarrow X_0 \; - - - - - \; X_{\leq n} \hookrightarrow X_{\leq n+1} \cdots \longrightarrow X$$

where $X_{\leq n}$ has only cells of height $\leq n$ – obtain

$$\begin{tikzcd} X_{n+1}\cdot \delta Y_{n+1} \arrow[r] \arrow[d] & X_{\leq n} \arrow[d] \\ \underbrace{X_{n+1}}_{\text{set of } (n+1)\text{-cells}}\cdot Y_{n+1} \arrow[r] & X_{\leq n+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

i.e. glue on the $(n+1)$-cells @ stage $n+1$.

Consider a globular theory $\mathbb{T}$:

$$\begin{tikzcd} \mathbb{G} \arrow[r, "()"] \arrow[rr, bend right, "D"'] & \Theta_0 \arrow[r, "J"] & \mathbb{T} \end{tikzcd}$$

Recall that $\mathbb{T}$ is contractible,

$\Leftrightarrow$ each glob. set $\mathbb{T}(D-, \bar{m})$ is contractible

We can view this as a functor

$$\begin{aligned} U_{\bar{m}} : \mathbb{G}\text{-Th} &\longrightarrow [\mathbb{G}^{op}, \mathrm{Set}] \\ \mathbb{T} &\longmapsto \mathbb{T}(D-, \bar{m}) \end{aligned}$$

Let us take for granted

(*) $\mathbb{G}$-Th is cocomplete,

each $U_{\bar{m}}$ has a left adjoint $F_{\bar{m}}$ preserving f.p. objects.

(These props hold much more generally - see eg. Monads & Theories (JB / R. Garner))

Assuming this, we can consider the set of maps in $\mathbb{G}$-Th:

$$B^* = \left\{ F_{\bar{m}}(\partial Y_n) \xrightarrow{F_{\bar{m}}(j_n)} F_{\bar{m}}(Y_n) : \bar{m} \in \Theta_0, n \in \mathbb{N} \right\}$$

all of which have f.p. domain

& now

<u>Theorem</u>

Π is a coherator iff it is B^*-cellular & B^*-contractible.

<u>Proof</u>

- B^*-contractibility of Π says

$$\begin{tikzcd} F_{\bar{m}}(\partial Y_n) \arrow[r, "\forall f"] \arrow[d, "F_{\bar{m}}(j_n)"'] & \Pi \\ F_{\bar{m}}(Y_n) \arrow[ur, dashed, "\exists g"'] & \end{tikzcd} \quad \text{or by } \underline{\text{adjointness}} \quad \begin{tikzcd} \partial Y_n \arrow[r, "\forall f"] \arrow[d, "j_n"'] & U_{\bar{m}}\Pi \\ Y_n \arrow[ur, dashed, "\exists g"'] & \end{tikzcd}$$

which says exactly that Π is contractible.

- B^*-cellularity says that $\exists$ colimit

$$\Theta_0 \to \Pi_0 \to \dots \to \Pi_j \to \Pi_{j+1} \to \dots \Pi$$

where $\Pi_j \to \Pi_{j+1}$ is a pushout of a coproduct of maps in B^*.

Let's just consider a pushout of a single map -

in fact to give a commutative square as below left

$$\begin{tikzcd} F_{\bar{m}}(\partial Y_n) \arrow[r] \arrow[d, "F_{\bar{m}}(j_n)"'] & \mathbb{R} \arrow[d, "K"] \\ F_{\bar{m}}(Y_n) \arrow[r] & \$ \end{tikzcd} \qquad \begin{tikzcd} \partial Y_n \arrow[r] \arrow[d, "j_n"'] & \mathbb{R}(D-, \bar{m}) \arrow[d, "K"] \\ Y_n \arrow[r] & \$(D-, \bar{m}) \end{tikzcd}$$

is (by adjointness) to give a square as above right -

The top map gives a parallel pair of $(n-1)$-cells $D_n \overset{f}{\underset{g}{\rightrightarrows}} \bar{m}$ a parallel pair of n-cells in $\bar{m}$

& the lower map provides a filler

$$\begin{tikzcd}
Dn \arrow[r, shift left, "Kf"] \arrow[r, shift right, "Kg"'] \arrow[d, shift right, "KD\sigma"'] \arrow[d, shift left, "KD\tau"] & \overline{m} \\
D(n+1) \arrow[ur, dashed, "\varphi_{f,g}"'] &
\end{tikzcd}$$

so the universal such square

$$\begin{tikzcd}
F_{\overline{m}}(\partial Y_n) \arrow[r, "\langle f,g\rangle"] \arrow[d, "F_{\overline{m}}(j_n)"'] & R \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\
F_{\overline{m}}(Y_n) \arrow[r] & R\langle f,g\rangle
\end{tikzcd}$$

has the universal property that it is obtained by freely adding a filler $\varphi_{f,g}$ for the parallel pair (f,g).

More generally, to say $\Pi_j \longrightarrow \Pi_{j+1}$ is a pushout of a coproduct of maps in $\mathcal{B}^*$ is to say that Π_{j+1} is obtained by freely adding fillers for a set of parallel pairs in Π_j. $\square$

Summary

- Contractibility ~ ∞-groupoid str
  - Cellularity ~ weakness (no strict equations)

Cellular contractible theories are the theories for weak ∞-groupoids.

These are called coherators.

The homotopy hypothesis made precise

- If Π is a coherator & $\mathbb{S}$ contractible, then $\exists\ \Pi \xrightarrow{K} \mathbb{S} \in \mathbb{G}\text{-Th}$:

since
$$\begin{tikzcd}
 & \Theta_0 = \emptyset \arrow[dl, "\in B^*\text{-cell}"'] \arrow[dr] & \\
\Pi \arrow[rr, dashed, "\exists"] \arrow[dr] & & \mathbb{S} \arrow[dl, "\in B^*\text{-Inj}"] \\
 & 1 &
\end{tikzcd}$$
&

this induces
$$\begin{tikzcd}
\mathrm{Mod}(\mathbb{S}) \arrow[rr, "K^*"] \arrow[dr, "U^{\mathbb{S}}"'] & \arrow[d, phantom, "="] & \mathrm{Mod}(\Pi) \arrow[dl, "U^{\Pi}"] \\
 & {[\mathbb{G}^{op}, \mathrm{Set}]} &
\end{tikzcd}$$
by restr.

Then K^* preserves weak equivalences since it preserves the construction of homotopy groups (exercise) or since, in fact, being a weak equivalence is a property of the

underlying map of globular sets.
In particular, if $\mathbb{T}$ is a coherator,

$$\mathbb{T} \xrightarrow{\exists K} \mathbb{T}_{Top}$$ – globular theory of top. spaces

& so

$$\begin{tikzcd} Top \arrow[r, "N_J"] \arrow[rr, bend left, "N_\infty"] & Mod(\mathbb{T}_{Top}) \arrow[r, "K^*"] & Mod(\mathbb{T}) \end{tikzcd}$$

preserves weak equivalences since both components do.

Grothendieck's homotopy hypothesis (precise form)

For any coherator $\mathbb{T}$,
the induced functor

$$Top(W^{-1}) \longrightarrow Mod(\mathbb{T})(W^{-1})$$

is an equivalence of categories.

Comments

① Formulated in 1983 by Grothendieck (Pursuing Stacks)

② Really one would like a bit more - to put a model structure on $\mathrm{Mod}(\Pi)$ for Π a coherator such that

$$\mathrm{Top} \xrightarrow{N_\infty} \mathrm{Mod}(\Pi)$$

is a Quillen equivalence.

And to prove that for any 2 coherators Π & Π', $\mathrm{Mod}(\Pi)$ & $\mathrm{Mod}(\Pi')$ are suitably equivalent.

Open questions.

③ Smaller question

Is the globular theory Π_{Top} cellular?

ie. is Π_{Top} a coherator?

(Update: no, it isn't - explained next week.)

Examples of coherators

- Both the small object argument & efficient small object argument applied to B^* provide examples of coherators

$$\phi = \Theta_0 \xrightarrow[B^*\text{-cellular}]{} \Pi \xrightarrow[B^*\text{-contractible}]{} 1$$

- The efficient soa produces the free B^*-algebraic injective on ϕ - that is, the initial object of $\mathrm{Alg}(B^*)$ - these are globular theories equipped with a contraction:

That is, a theory Π equipped with:

For each parallel pair $n \overset{f}{\underset{g}{\rightrightarrows}} \bar{m} \in \Pi$ we are given a chosen lifting

$$\begin{tikzcd} n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift left] \arrow[d, shift right] & \bar{m} \\ n+1 \arrow[ur, "\varphi(f,g)"'] & \end{tikzcd}$$

This does not follow from the theorem last week on the efficient soa immediately, since I don't expect

$$\mathrm{cell}(B^*) \subseteq \mathrm{Mono}.$$

- However it is true that if Π is cellular, then each cellular map

(*) $\Pi \longrightarrow \mathcal{S}$ is mono (i.e. id on obs & faithful).

- This is all that is needed to show that the efficient soa produces free algebraic injectives on cellular objects

in particular since $\phi = \Theta_0$ is cellular, the efficient soa applied to $\Theta_0 \longrightarrow 1$ produces the initial globular theory with contraction.

(*) above a bit technical to prove - JB "iterated algebraic injectivity..."

Lecture 6 - Grothendieck ω-groupoids from Type Theory & weak factorisation systems

This time:
how to get an internal Gr. ω-groupoid str on a topological space, a type, a Kan complex ...

The idea:
If $\mathcal{C}$ has a weak fact. system $(\mathcal{L}, \mathcal{R})$ then can form path object on X:

$$\begin{tikzcd} X \arrow[r, "i \in \mathcal{L}"] \arrow[dr] & PX \arrow[d, "\langle s,t \rangle \in \mathcal{R}"] \\ & X \times X \end{tikzcd}$$

Writing $X_0 = X$, $X_1 = PX$, this is the start of a globular object.

- Now paths are supposed to be composable so we ought to have

$$X_1 \times_{X_0} X_1 \xrightarrow{comp} X_1 .$$

It should be a filler for the

square

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d, "\langle i,i \rangle"'] & X_1 \arrow[d, "{\langle s,t \rangle \in \mathcal{R}}"] \\ X_1 \times_{X_0} X_1 \arrow[ur, "\mathrm{comp}"] \arrow[r, "{\langle s,t \rangle}"'] & X_0 \times X_0 \end{tikzcd}$$

if we want the composite of id. paths to be an id.

If the left vertical is $\in \mathcal{L}$ - we get such a composition by the lifting prop. & this is the start of an ∞-groupoid str, involving higher paths ...

Remark: Asking such maps are in $\mathcal{L}$ is closely related to homotopy type theory where if you can define an operation on identity paths you can do it everywhere (path induction)

- This reminds me of something I once knew!
- For topological spaces, the identity path on $x \in X$ is the constant path $\Delta x : [0,1] \longrightarrow X$.
- The composite of constant paths in Top is constant!
- This indicates that the theory Π_{Top} is not cellular: indeed we have an equation

$$\begin{tikzcd} D1 \arrow[r, "const"] \arrow[d, "comp"'] & D0 \\ D(1,0,1) \arrow[ur, "{\langle const, const \rangle}"'] & \end{tikzcd}$$

in Π_{Top}.

- Indeed in Π_{Top}, $D0 = (0)$ is terminal!!

FACT! : If $(0) \in \Pi$ is terminal, Π is not cellular.

Proof : Still to formalise it properly.

Identity type categories

An identity-type category is a cat $\mathcal{C}$ equipped w/ a weak factorisation system $(\mathcal{L}, \mathcal{R})$ such that

- A terminal ob 1 exists & each $!: X \to 1 \in \mathcal{R}$
- Pullbacks of $\mathcal{R}$-maps exist & the pullback of an $\mathcal{L}$-map along an $\mathcal{R}$-map is an $\mathcal{L}$-map.

Remark) • The pullback of an $\mathcal{R}$-map is of course an $\mathcal{R}$-map (true for any wfs).

Examples

Concept introduced by Gambino-Garner (The identity type wfs). Other authors have considered related structures (Joyal - Tribes), Shulman (Type-theoretic fibration cats).

- Main point :- syntactic category of dependent type theory comes equipped w/
  - class of fibrations $\mathcal{R}$.
- Fibrations $B \to A$ correspond to dependent types over the base $x \in A \vdash B(x)$ type.

More accurately, for each dependent type B as above, we have a dependent projection

$$p : (x : A, y : B(x)) \longrightarrow (x : A).$$

- If we add "identity types", we get a wfs satisfying the above axioms.

Id types give ① $x, y : A \vdash \mathrm{Id}_A(x,y) : \mathrm{Type}$
& ② $r(x) := \mathrm{refl}(x) \in \mathrm{Id}_A(x,x)$.

- i.e. a factorisation in $\mathcal{S}$ of diagonal.

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[dr, "\Delta"'] & \mathrm{Id}_A \arrow[d, "p"] \\ & A^2 \end{tikzcd}$$

- The rules for identity types say

③ given $x, y : A, z : \mathrm{Id}_A(x,y) \vdash C(x,y,z) \vdash \mathrm{Type}$
& $c(x) : C(x,x,r(x))$

then ④ have $J_c(x,y,u) : C(x,y,u)$
⑤ $J_c(x,x,r(x)) = c(x)$

These correspond to

③ $$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "r"'] & C \arrow[d] \\ \mathrm{Id}_A \arrow[r, equal] & \mathrm{Id}_A \end{tikzcd}$$ & then $$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "r"'] & C \arrow[d] \\ \mathrm{Id}_A \arrow[r, equal] \arrow[ur, "J"] & \mathrm{Id}_A \end{tikzcd}$$

⑤ ④

so the reflexivity maps have left lifting prop wrt dependent projections.

- To make above precise, I point out that the objects of syntactic category are dependent contexts like

$(x : A, y : A, z : \mathrm{Id}_A(x,y))$ for instance.

(2) Lots of other examples:

- cat of Kan complexes
- top spaces

Reflexive globular contexts

- Given $X \in \mathcal{C}$ an ids. type category, we can extend it to a globular object $X : \mathbb{G}^{op} \to \mathcal{C}$.
- We set $X(0) = X$ & define $X(1)$ by the factorisation

$$\begin{tikzcd}
X(0) \arrow[r, "i_{0,1} \in \mathcal{L}"] \arrow[dr, "\Delta"'] & X(1) \arrow[d, "\langle s_1, t_1 \rangle \in \mathcal{R}"] \\
 & X(0) \times X(0)
\end{tikzcd}$$

- This makes $X(1) \overset{s_1}{\underset{t_1}{\rightrightarrows}} X(0)$ a graph or 1-globular object.
- We extend it inductively:

  suppose we have constructed X as an n-globular object, we must show how to extend it to a (n+1)-globular object.

- Given a n-globular object X, can define m-boundary $B_m X$ of X for $1 \leq m \leq n+1$.

- Its universal property is that
$$\mathcal{C}(A, B_{m+1} X) \cong \text{Parallelpairs in } \mathcal{C}(A, X-) \text{ of } m\text{-cells}.$$

- Therefore

$$\begin{tikzcd}
A \arrow[dr, "{\langle f,g \rangle}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& B_{m+1}X \arrow[r, "p_m"] \arrow[d, "q_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{\langle s,t \rangle}"] \\
& X_m \arrow[r, "{\langle s,t \rangle}"'] & (X_{m-1})^2
\end{tikzcd}$$

- This def^n is the clearest one but observe we have

$$\begin{tikzcd}
X_m \arrow[r, "{\exists \langle s,t \rangle}"] \arrow[dr, "{\langle s,t \rangle}"'] & B_m X \arrow[d, tail, "{\langle p_{m-1}, q_{m-1} \rangle}"] \\
& (X_{m-1})^2
\end{tikzcd}$$

& then

$$\begin{tikzcd}
B_{m+1}X \arrow[r, "p_m"] \arrow[d, "q_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "{\langle s,t \rangle}"] \\
X_m \arrow[r, "{\langle s,t \rangle}"'] & B_m X
\end{tikzcd}$$

equivalently since $\langle p_m, q_m \rangle$ is monic.

- The inductive construction now works as: suppose we have the n-glob. object X & suppose that $B_m X$ exists &

$$X_m \xrightarrow[\langle s,t\rangle]{} B_m X \in \mathcal{R} \quad \forall m \leq n.$$

(Note this is true for $n=1$ since

$$X_1 \xrightarrow{\langle s,t\rangle} B_1 X = X_0 \times X_0 \in \mathcal{R}.\ )$$

- Then the pullback

$$\begin{tikzcd} B_{m+1}X \arrow[r, "p_m \in \mathcal{R}"] \arrow[d, "\mathcal{R} \ni q_m"'] \arrow[dr, phantom, "\lrcorner"] & X_m \arrow[d, "\langle s,t\rangle \in \mathcal{R}"] \\ X_m \arrow[r, "\langle s,t\rangle \in \mathcal{R}"'] & B_m X \end{tikzcd}$$

exists

& we have the induced map

$$\begin{tikzcd} X(n) \arrow[dr, "\langle 1,1\rangle"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\ & B_{n+1}X \arrow[r, "p_n"] \arrow[d, "q_n"'] \arrow[dr, phantom, "\lrcorner"] & X(n) \arrow[d, "\langle s,t\rangle"] \\ & X(n) \arrow[r, "\langle s,t\rangle"'] & B_n X \end{tikzcd}$$

& define $X(n) \xrightarrow{i_{n,n+1}} X(n+1)$ $\in \mathcal{L}$

$$\begin{tikzcd} X(n) \arrow[r, "i_{n,n+1}"] \arrow[dr, "\langle 1,1\rangle"'] & X(n+1) \arrow[d, "\langle s_{n+1}, t_{n+1}\rangle"] \\ & B_{n+1}X \end{tikzcd}$$

$\in \mathcal{R}$

& now $X(n+1) \underset{t_{n+1}}{\overset{s_{n+1}}{\rightrightarrows}} X(n)$ extends X to a (n+1)-globular object.

By induction, we obtain a globular object X satisfying

① $\exists$ $\mathcal{L}$-maps $i_{n,n+1} : X(n) \longrightarrow X(n+1)$ making X a reflexive globular object.

② The maps $X(n+1) \underset{t_n}{\overset{s_n}{\rightrightarrows}} X(n)$ &

$\langle s_n, t_n\rangle : X(n+1) \longrightarrow B_{n+1}X$ are $\mathcal{R}$-maps.

A globular object with these props will be called a reflexive globular context.

Theorem

Any reflexive glob. context X ad. the structure of Groth ∞-groupoid.

We will construct it using endomorphism theories.

Endomorphism Theories

- Let $G^{op} \xrightarrow{A} \mathcal{C}$ be a globular object.
- If $\mathcal{C}$ has A-glob. products, can right Kan extend

$$
\begin{tikzcd}
\Theta_0^{op} \arrow[dr, "A(-)"] & \\
G^{op} \arrow[u, "D^{op}"] \arrow[r, "A"'] \arrow[ur, phantom, "\cong"] & \mathcal{C}
\end{tikzcd}
$$

~ globular product preserving functor.

- Now factor

$$
\begin{tikzcd}
\Theta_0^{op} \arrow[r, "J_A^{op}"] \arrow[dr, "A(-)"] & (\mathrm{End}A)^{op} \arrow[d, "K_A"] \\
G^{op} \arrow[u, "D^{op}"] \arrow[r, "A"'] \arrow[ur, phantom, "\cong"] & \mathcal{C}
\end{tikzcd}
$$

id. on obs

f.f.

- Both J_A^{op} & K_A preserve globular products by construction.
- Then $\Theta_0 \xrightarrow{J_A} \mathrm{End}(A)$ pres glob. sums, so it is a globular theory, the endomorphism theory of A.
- Explicitly, $\mathrm{End}(A)(\bar{n}, \bar{m}) = \mathcal{C}(A(\bar{m}), A(\bar{n}))$
- In particular, $K_A : (\mathrm{End}A)^{op} \longrightarrow \mathcal{C}$, $\bar{n} \longmapsto A(\bar{m})$

equips A with structure of End A-model.

Exercise

There is a natural bijection

$\mathbb{G}\text{-Th}(\mathbb{T}, \mathrm{End}A) \cong (\mathrm{Mod}\,\mathbb{T})_A$ – the set of $\mathbb{T}$-model structures on A.

It is obtained by postcomposing by K_A.

In particular, a good way to put Groth. ∞-groupoid structure on $A : \mathbb{G}^{op} \to \mathcal{C}$ is to show $\mathrm{End}A$ is contractible: then the $\mathrm{End}A$-model structure on A exhibits A as an ω-groupoid.

Now $\mathrm{End}A$ is contractible just when for each $\bar{m} \in \Theta_0$, the globular set

$$\mathrm{End}A(JD-, \bar{m}) \text{ is contractible.}$$

But $\mathrm{End}A(JDn, \bar{m}) = \mathcal{C}(A\bar{m}, An)$

so this just says that

- each $\mathcal{C}(A\bar{m}, A-) : \mathbb{G}^{op} \to \mathbf{Set}$ is a contractible globular set:

In el. terms, given $A\bar{m} \overset{f}{\underset{g}{\rightrightarrows}} An$
st $sf = sg$ & $tf = tg$ (f & g are parallel)

$$
\begin{tikzcd}
 & A(n+1) \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
A\bar{m} \arrow[ur, "\exists"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & An
\end{tikzcd}
$$

This is how we will construct an ω-groupoid from type theory.

Theorem

Any reflexive glob. context A ad. the structure of Groth ∞-groupoid.

~~Proof~~ - Let us write $A(n) \xrightarrow{i_{n,m}} A(m)$ for $m>n$ for the maps obtained by composing the $i_{n,n+1}$'s,

& $A(0) \xrightarrow{i_m := i_{0,m}} A(m)$.

- These give a cone $\Delta A(0) \xrightarrow{i} A \in [G^{op}, \mathcal{C}]$

since

$$\begin{tikzcd}
 & A(n) \arrow[d, shift right, "s_n"'] \arrow[d, shift left, "t_n"] \\
A(0) \arrow[ur, "i_n"] \arrow[r, "i_{n-1}"'] \arrow[ur, phantom, "=", near end, shift right=3] & A(n-1)
\end{tikzcd}$$

- So we obtain

$$n \longmapsto A(0) \xrightarrow{i_n} A(n)$$

$$\begin{tikzcd}
G^{op} \arrow[r, "i/A"] \arrow[dr, "A"'] \arrow[dr, phantom, "=", shift left=3] & A(0)/\mathcal{C} \arrow[d, "cod"] \\
 & \mathcal{C}
\end{tikzcd}
\qquad
\begin{array}{c} [\,\cdot\,] \\ \downarrow \\ A(n) \end{array}$$

- We will prove that

(1) $A(0)/\mathcal{C}$ has i/A-glob. products

② $\mathrm{End}(i/A)$ is contractible.

Then the composite

$$\mathrm{End}(i/A) \xrightarrow{K_{i/A}} A(0)/\mathcal{C} \xrightarrow{\mathrm{cod}} \mathcal{C}$$

pres. glob. prods

pres all connected lims

preserves globular products & so will exhibit $\mathrm{End}(i/A)$-model str. on A - ie. ω-groupoid structure.

First, we prove ①.

In constructing globular products in $\mathcal{C}$, we need to construct them using iterated pullbacks, since $\mathcal{C}$ only has some pullbacks:

eg.

$$\begin{tikzcd}
& & & A(n_1,n_2,n_3,n_4,n_5) \arrow[dll] \arrow[ddr] \arrow[ddl, phantom, "\lrcorner", very near start] & \\
& A(n_1,n_2,n_3) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & & \\
A(n_1) \arrow[dr, "t"'] & & A(n_3) \arrow[dl, "s"] \arrow[dr, "t"'] & & A(n_5) \arrow[dl, "s"] \\
& A(n_2) & & A(n_4) &
\end{tikzcd}$$

Use induction over length of tod $\bar{n} = (n_1, \ldots, n_k)$.

- We will write $A(\bar{n}) \xrightarrow[p_j^{\bar{n}}]{} A(n_j)$ for the limit projection,

& $A(0) \xrightarrow{i_{\bar{n}}} A(\bar{n})$ for glob. prod in $X(0)/\mathcal{C}$ which satisfies

$$\begin{tikzcd} A(0) \arrow[r, "i_{\bar{n}}"] \arrow[dr, "i_j"'] & A(\bar{n}) \arrow[d, "p_j^{\bar{n}}", "\perp"'] \\ & A(n_j) \end{tikzcd}$$

- Certainly $A(0)/\mathcal{C}$ has glob. products for tod's of length 1.

- For $\bar{n}^+ = (n_1, \ldots, n_k, n_{k+1}, n_{k+2})$, the glob prod is the pullback

$$\begin{tikzcd}[column sep=large] A(\bar{n}^+) \arrow[rr, "p_{k+2}^{\bar{n}^+}"] \arrow[d, "q"', "\in R"] \arrow[dr, phantom, "\lrcorner", very near start] & & A(n_{k+2}) \arrow[d, "\perp"', "s \in R"] \\ A(\bar{n}) \arrow[r, "p_k^{\bar{n}}"'] & A(n_k) \arrow[r, "t \in R"'] & A(n_{k+1}) \end{tikzcd}$$

which exists since $s \in R$.

- The glob. product in $A(0)/\mathcal{C}$ is then the unique map

$$\begin{tikzcd}
A(0) \arrow[dr, "i_{\bar n^+}"] \arrow[drrr, bend left, "i_{n_{k+2}}"] \arrow[ddr, bend right, "i_{\bar n}"'] & & & \\
& A(\bar n^+) \arrow[rr, "p^{\bar n^+}_{k+2}"] \arrow[d, "q"', "\in R"] \arrow[drr, phantom, "\lrcorner", very near start] & & A(n_{k+2}) \arrow[d, "s \in R"] \\
& A(\bar n) \arrow[r, "p^{\bar n}_k"'] & A(n_k) \arrow[r, "t \in R"'] & A(n_{k+1})
\end{tikzcd}$$

to the pullback.

- Our inductive construction also proves: each final projⁿ $p^{\bar n}_k \in R$.

- Indeed, if $p^{\bar n}_k \in R$, so is the lower leg in diagram. Hence so is the upper leg by pullback stability.

- We also will prove by induction that each $i_{\bar{n}} : A(0) \longrightarrow A(\bar{n}) \in \mathcal{L}$.

- Since the right leg s is split epi, so is its pullback q – indeed:

$$\begin{tikzcd}
A(\bar{n}) \arrow[r, "p_k^{\bar{n}}"] \arrow[d, dashed, "\exists! i'"] \arrow[dd, bend right, "1"'] & A(n_k) \arrow[r, "t"] & A(n_{k+1}) \arrow[d, "i"] \arrow[dd, bend left, "1"] \\
A(\bar{n}^+) \arrow[rr, "p_{k+2}^{\bar{n}^+}"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & & A(n_{k+2}) \arrow[d, "s"] \\
A(\bar{n}) \arrow[r, "p_k^{\bar{n}}"'] & A(n_k) \arrow[r, "t"'] & A(n_{k+1})
\end{tikzcd}$$

Since lower square & outer square are pullbacks, so is upper. As $i \in \mathcal{L}$, so is $i' \in \mathcal{L}$ as pullbacks of $\mathcal{L}$-maps along $\mathcal{R}$-maps are $\mathcal{L}$-maps.

- Remains to show

$$\begin{tikzcd} A(0) \arrow[r, "i_{\bar{n}} \in \alpha"] \arrow[drr, "i_{\bar{n}^+}"', "\in \alpha"'] & A(\bar{n}) \arrow[dr, "i' \in \alpha"] & \\ & {} \arrow[u, phantom, "="] & A(\bar{n}^+) \end{tikzcd}$$

since the claim then follows by induction.

- Both give $i_{\bar{n}}$ when postcomposed by pullback projection q.

- Certainly

$$p^{\bar{n}^+}_{k+2} \circ i_{\bar{n}^+} = in_{k+2} .$$

Also

$$p^{\bar{n}^+}_{k+2} \circ i' \circ i_{\bar{n}} \overset{\text{def}}{=} i \circ t \circ p^{\bar{n}}_{k} \circ i_{\bar{n}}$$

$$\underset{\substack{\text{def of} \\ i_{\bar{n}} \text{ as limit}}}{=} i \circ t \circ in_k \underset{\text{cone}}{=} i \circ in_{k+1} \underset{\text{def}}{=} in_{k+2}$$

Hence they agree on postcomp. w' the pullback projections

It remains to prove contractibility of $End(i/A)$.

- This amounts to showing that given

$$\begin{tikzcd}
A(0) \arrow[d, "i_{\bar n}"'] \arrow[drr, "i_m"] & & \\
A(\bar n) \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & A(m)
\end{tikzcd}
\qquad \text{then} \qquad
\begin{tikzcd}
A(\bar n) \arrow[r, "\exists h"] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & A(m+1) \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
& A(m)
\end{tikzcd}$$

s.t. $s \circ f = s \circ g$ &
$t \circ f = t \circ g$

- Get $\langle f, g \rangle : A(\bar n) \longrightarrow B_{m+1}A$ &

then

$$\begin{tikzcd}
A(0) \arrow[r, "i_{m+1}"] \arrow[d, "\alpha \ni i_{\bar m}"'] & A(m+1) \arrow[d, "\langle s,t \rangle"', "\in R"] \arrow[ddr, bend left, "s"] \arrow[ddr, bend left=15, "t"'] & \\
A(\bar m) \arrow[r, "\langle f,g \rangle"'] \arrow[ur, dashed, "\exists h"] \arrow[drr, bend right, "f"] \arrow[drr, bend right=40, "g"'] & B_{m+1}A \arrow[dr, shift left, "p_m"] \arrow[dr, shift right, "q_m"'] & \\
& & A(m)
\end{tikzcd}$$

The square commutes since p_m, q_m are jointly mono. Hence we obtain a diagonal filler completing the proof. $\square$

References

- This was proven by Van den Berg & Garner "Types are weak ω-groupoids" using Batanin weak ω-cats.
- I wrote a short expository paper showing how their proof can be done much more briefly if we use Grothendieck ω-groupoids,

Note on the construction of globular weak ω-groupoids from types, topological spaces etc.

which is what this lecture was based on.

Higher cats course part 2 - simplicial higher categories

Post-course summary:

This was a quick overview to some simplicial approaches to (∞,n)-categories covering -

L7) ∞-groupoids & $(\infty,1)$-cats : Kan complexes & quasicats.

L8) Other models of $(\infty,1)$-cats : - simpl. enriched cats
- Segal cats
- complete Segal spaces

L9) ∞-cosmoi : model independent approach to $(\infty,1)$-cats

L10 & L11) Simp models of $(\infty,2)$-cat & higher, covering higher quasicats, higher complete Segal spaces & complicial sets.

Note) Some of this stuff I was just learning on the fly so could no doubt be improved, and any comments very welcome.

L7 - Kan complexes & quasicategories

- Kan complexes $\equiv$ simplicial ∞-groupoids
- Quasicats $\equiv$ simplicial $(\infty,1)$-categories,
  meaning all n-morphisms above dimension 1 are invertible.
  Also called ∞-categories.
- So Kan complexes $\equiv$ simplicial $(\infty,0)$-cats
  quasicats $\equiv$ - - - - $(\infty,1)$-cats
  ? $\equiv$ - - -

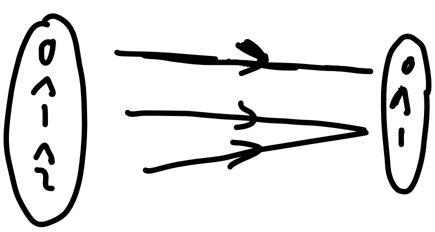
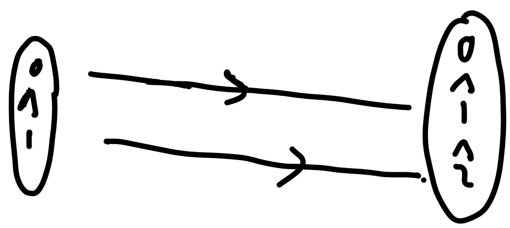
- Both are defined as simplicial sets with properties.

- Here Δ = simplicial cat of non-empty finite ordinals
  $[n] = \{0 < \ldots < n\}$ for $n \geq 0$
  & order preserving maps.

The factorisation system

Δ has a strict fact. system (Surj / Inj).

- The <u>surjections</u> are generated by the maps
  $$\sigma_i^n : [n+1] \twoheadrightarrow [n] \quad \text{for } 0 \le i \le n$$
  taking value @ i twice;
- The <u>injections</u> are generated by the maps
  $$\delta_i^n : [n-1] \rightarrowtail [n] \quad \text{for } 0 \le i \le n$$
  which omit i.

$\sigma_1^1 : [2] \to [1]$	$\delta_0^2 : [1] \longrightarrow [2]$
	

E.g

- Δ is freely generated by these maps subject to the <u>simplicial identities</u>:
  - comp. of two δ's
  - comp. of two σ's
  - Rewrite $\sigma . \delta$

  which I won't use.

Simplicial sets

- Write sSet for $[\Delta^{op}, \mathrm{Set}]$, the cat. of simplicial sets.
- A simplicial set X comes equipped with
$$X_n \xrightarrow{d_n^i} X_{n-1}, \quad X_n \xrightarrow{s_n^i} X_{n+1}$$
for $0 \leq i \leq n$.
- Elements of X_n called n-simplices.
- In low dimensions, have

$$\begin{tikzcd}[column sep=large]
\cdots X_2 \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & X_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=4, "d_1^0"] \arrow[r, shift right=4, "d_1^1"'] & X_0 \arrow[l, "s_1^0"]
\end{tikzcd}$$

$$\begin{tikzcd}
& b \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\
a \arrow[ur, "f"] \arrow[rr, "h"'] & {} & c
\end{tikzcd} \qquad\qquad a \xrightarrow{f} b \qquad\qquad a, b$$

No joy drawing beyond X_3!

Yoneda embedding

$$y : \Delta \longrightarrow \mathrm{SSet}$$

$$n \longmapsto \Delta^n, \text{ the } n\text{-simplex.}$$

Δ^0	$\bullet$
Δ^1	$0 \longrightarrow 1$
Δ^2	$\begin{tikzcd}[row sep=small] & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$

+ degenerate higher cells

Just write

$$n \xrightarrow{F} m \longmapsto \Delta^n \xrightarrow{f} \Delta^m$$

Two embeddings

- We have the full inclusion

$$J : \Delta \longrightarrow Cat .$$

indeed, we saw it earlier as the inclusion of the graphical theory of categories.

- Also

$$\Delta \xrightarrow{K} Top$$

$$[n] \longmapsto |\Delta^n|$$

$$= \{ (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \mid \Sigma x_i \leq 1 \}$$

views $[n]$ as standard n-simplex $|\Delta^n|$,

& $f : [n] \to [m] \longmapsto |\Delta^n| \xrightarrow{|f|} |\Delta^m|$

$$|f|(x_1, \dots, x_n) = (y_0, \dots, y_m)$$

where $y_j = \sum_{i \in f^{-1}(j)} x_i$

These induce

(*)
$$
\begin{tikzcd}
 & \mathrm{SSet} & \\
 & & \\
\mathrm{Top} \arrow[uur, "N_K"] & \Delta \arrow[l, "K"'] \arrow[r, "J"'] \arrow[uu, "y"] & \mathrm{Cat} \arrow[uul, "N_J"']
\end{tikzcd}
$$

- Here $N_K X := \mathrm{Sing}(X)$ is the singular complex of X, with value $\mathrm{Sing}X_n = \mathrm{Top}(|\Delta^n|, X)$ the set of n-simplices in X.

- $N_J C := NC$ is the nerve of C:

$NC_n = \mathrm{Cat}([n], C)$

$= \{$composable sequences of length n$\{$

$$\begin{tikzcd}
 & & \mathrm{SSet} \arrow[dll, shift right, "|-|"'] \arrow[drr, shift left, "h"] \arrow[dll, phantom, "\perp"] \arrow[drr, phantom, "\perp"] & & \\
\mathrm{Top} \arrow[urr, shift right, "N_K"'] & & \Delta \arrow[ll, "K"] \arrow[rr, "J"'] \arrow[u, "y"] & & \mathrm{Cat} \arrow[ull, shift left, "N_J"]
\end{tikzcd}$$

- Since SSet is the free cocompletion of Δ both of these have left adjoints, as depicted.
- $|X|$ is the geometric realisation of X, $|X| = \int^{[n]\in\Delta} X_n . |\Delta_n|$
- hX is category w/ obs X_0, arrows generated by 1-simplices $x \xrightarrow{f} y \in X_1$ subject to relations:
  - $x \xrightarrow{s_i(x)} x = id_x$
  - $$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\alpha"] & & z
\end{tikzcd} \in X_2 \Rightarrow g \circ f = h \in hX.$$

$(*)$

$$\begin{tikzcd}
& SSet \arrow[dl, shift right=1.5, "|-|"'] \arrow[dr, shift left=1.5, "h"] & \\
Top \arrow[ur, shift right=1.5, "Sing"'] \arrow[ur, phantom, "\perp"] & \Delta \arrow[l, "K"] \arrow[r, "J"'] \arrow[u, "y"] & Cat \arrow[ul, shift left=1.5, "N"] \arrow[ul, phantom, "\perp"]
\end{tikzcd}$$

- Certainly topological spaces give rise to ∞-groupoids (as we know) so $Sing X$ should be a simplicial ∞-groupoid.
- Likewise NC should be an ∞-category.
- As such, ∞-cats should be a common generalisation of $Sing X$ & NC - so lets explore some of their common properties.

Images, boundaries & horns

- Consider a set of maps $\{f_i : A_i \to B : i \in I\}$ in SSet.
- Factoring

$$\begin{tikzcd}
\sum_{i\in I} A_i \arrow[rr, "\Sigma f_i"] \arrow[dr, two heads] & & B \\
& \mathrm{im}(f_I) \arrow[ur, hook] &
\end{tikzcd}$$

as (pointwise surj / pointwise mono) produces the image of the f_i –

- explicitly $\mathrm{im}(f_I)[n]$ consists of those $x \in B[n]$ which are in the image of some $(f_i)_n : A_i(n) \to B(n)$

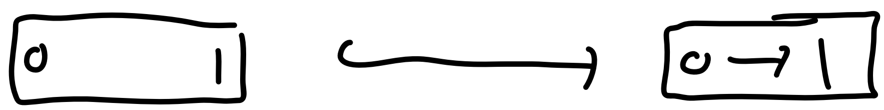
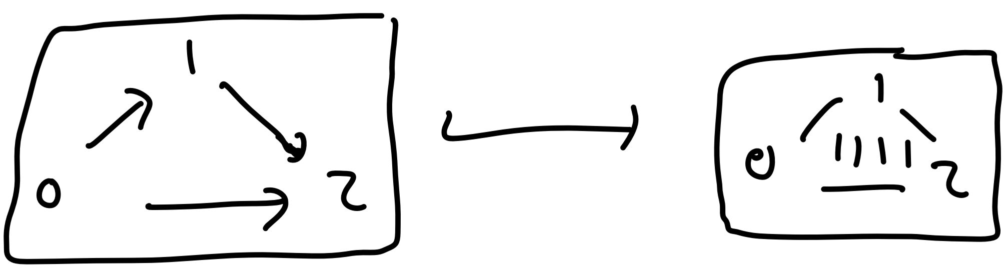
Examples

① The joint image of

$$\{\delta_i^n : \Delta^{n-1} \longrightarrow \Delta^n : 0 \le i \le n\}$$

produces $\partial\Delta^n \hookrightarrow \Delta^n$,

the boundary of the n-simplex.

$n=0$	$\emptyset \hookrightarrow \{\bullet\}$
$n=1$	
$n=2$	

② The joint image of the maps
$\sigma_i^n : \Delta^{n-1} \to \Delta^n$ for $i \neq k$, $n \geq 1$
produces the k'th horn inclusion

$$\Lambda^n_k \hookrightarrow \Delta^n$$

It is called an

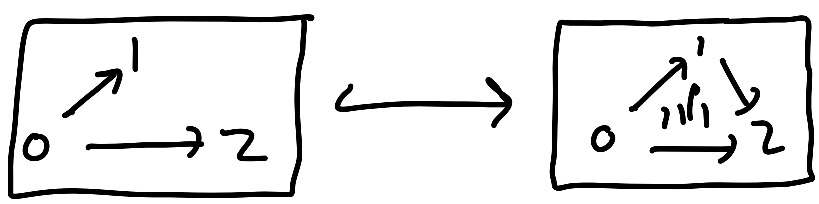
- outer horn if $k = 0$ or n
- inner horn if $0 < k < n$.

$n=1$ $\quad [0] \hookrightarrow [0 \to 1]$, $[1] \hookrightarrow [0 \to 1]$

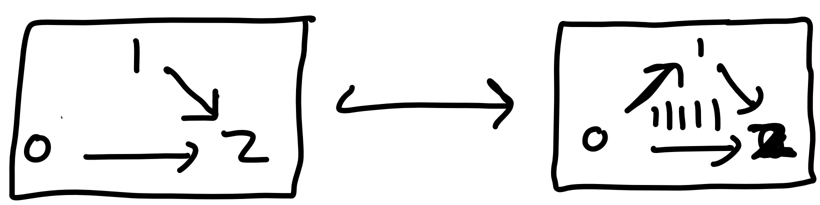
both outer.

$n=2$

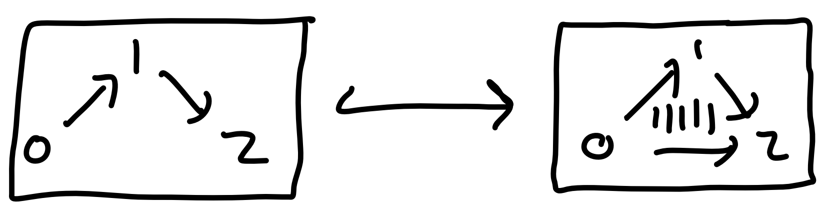
Outer



Outer



Inner



③ For $[1] \xrightarrow{\theta_{i,i+1}} [n]$ & $n \geq 1$

$0<1 \longmapsto i<i+1$

the joint image of the

$\theta_{i,i+1} \quad \Delta^1 \longrightarrow \Delta^n$

produces the spine of Δ^n:

$$Sp\,\Delta^n \hookrightarrow \Delta^n,$$

which just looks like

$$\boxed{0 \to 1 \to 2 \to 3 \to \cdots \to n}$$

with no 1-simplices $n \to m$ unless $m = n+1$.

Note: The inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ & $\Lambda^n_k \hookrightarrow \Delta^n$ completely determine the classical model structure on sSet: they are the gen. cofibrations & trivial cofibrations.

Two embeddings

- We have the full inclusion

$$J : \Delta \longrightarrow \mathrm{Cat} \quad .$$

indeed, we saw it earlier as the inclusion of the graphical theory of categories.

- Also

$$\Delta \xrightarrow{\quad K \quad} \mathrm{Top}$$

$$[n] \longmapsto |\Delta^n|$$

$$= \{(x_1,\dots,x_n) \in \mathbb{R}^n_{\geq 0} \mid \Sigma x_i \leq 1\}$$

views $[n]$ as standard n-simplex $|\Delta^n|$,

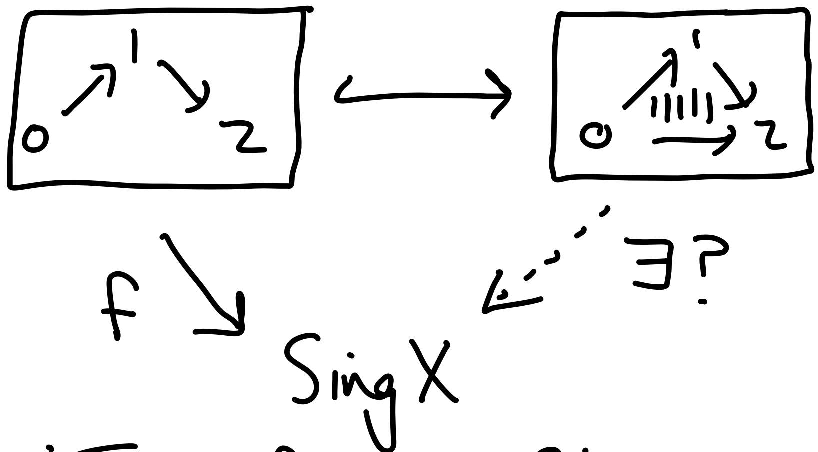
$$\& \quad f : [n] \to [m] \longmapsto |\Delta^n| \xrightarrow{|f|} |\Delta^m|$$

$$|f|(x_1,\dots,x_n) = (y_0,\dots,y_m)$$

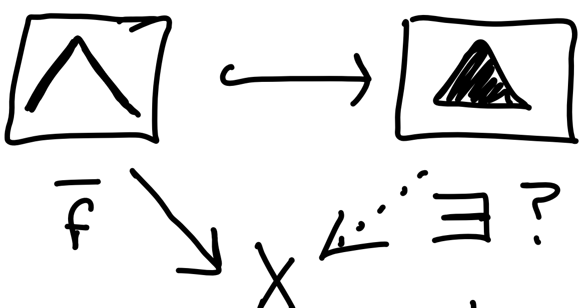
where $y_j = \sum_{i \in f^{-1}(j)} x_i$

Properties of Sing X

- Consider the inner horn



- By adjointness for $|-| \dashv \mathrm{Sing}$, this is to give



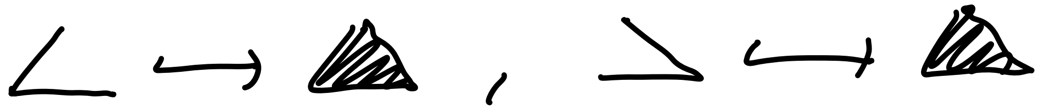
& we can find a filler by composing paths.

- In fact, the horizontal map is a deformation retraction (in particular a split mono) & has section s - then $\bar{f} \circ s$ is filler.

- Similarly all horn inclusions

$$\Lambda^n_k \hookrightarrow \Delta^n \in SSet$$

are sent by $|-| : SSet \longrightarrow Top$ to deformation retractions.

- E.g 

  - Hence $Sing X$ is injective to each horn inclusion.

Def^n A simplicial set X is a Kan-complex (aka ∞-groupoid) if $X \in Inj(\text{Horn Inclusions})$.

Properties of NC

- A map $\Lambda^2_1 = \boxed{\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}} \longrightarrow NC$

  picks out a pair $a \xrightarrow{f} b \xrightarrow{g} c \in C$.

- Unique extension $\Delta^2 \longrightarrow NC$ picking out 2-simplex
$$\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ a \arrow[ur, "f"] \arrow[rr, "gf"'] & {} & c \end{tikzcd}$$
in NC.

- Not true for outer horns

$$\boxed{\begin{tikzcd} & 1 \\ 0 \arrow[ur] \arrow[r] & 2 \end{tikzcd}} \quad \& \quad \boxed{\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}}$$

unless X a groupoid.

unique lifts

<u>Proposition</u>

Each NC is orthogonal to each inner horn inclusion.

~~Proof~~ - We have checked the case Λ^1_2 above.

- It suffices to check each
$$\Lambda^n_k \longrightarrow \Delta^n$$ is inverted by
$$h : SSet \longrightarrow Cat.$$
- Now h determined by 2-truncation & for $n \geq 4$, these have same 2-truncation - only omit some $(n-1)$-cells.
- Remains to consider Λ^3_1, Λ^3_2.

Λ^1_3 looks like

- So we see $h\Lambda^3_1$ = free cat gen by

$$0 \to 1 \to 2 \to 3,$$

ie. $n = h\Delta^n$, as required.

- Case Λ^3_1 similar. $\square$

Definition) A simplicial set X is a quasicategory (aka ∞-cat)

$$X \in \mathrm{Inj}(\text{Inner Homs}).$$

Corollary) Both SX & NC are ∞-cats.

Quasicategories - basic properties

- Let X be an ∞-cat.
- Its 0/1-simplices we call objects & morphisms.
- Given $a \xrightarrow{f} b \xrightarrow{g} c \in X$, we say $h : a \to c$ is a composite of g & f if $\exists$ 2-simplex
$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c \end{tikzcd}$$
in X.
- We then say that the triangle
$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c \end{tikzcd}$$
commutes & write $gf \sim h$.
- By filling Λ^2_1, each composable pair has a composite.
- Composites are not unique, but they are unique up to homotopy.
- Also write $a \xrightarrow{1} a$ for $s_0^1(a)$ - this will be identity.

Def^n) $a \underset{g}{\overset{f}{\rightrightarrows}} b \in X$ are htpic ($f \simeq g$) if any of

① $f \circ 1_a \sim g$ ② $1_b \circ f \sim g$

③ $g \circ 1_a \sim f$ ④ $1_b \circ g \sim f$.

Prop^n) The above 4 relations are equivalent in an ∞-cat X & an equivalence relation.

Proof

- First observe that in an ∞-cat X, given a diagram

$$
\begin{tikzcd}
 & b \arrow[rr, "j"] \arrow[dr, "g"] \arrow[d, phantom, "\circ"] & \arrow[d, phantom, "\circ"] & d \\
a \arrow[ur, "f"] \arrow[rr, "h"'] & \; & c \arrow[ur, "i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & b \arrow[rr, "j"] & \arrow[d, phantom, "A"] & d \\
a \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[urrr, "k"] & & c \arrow[ur, "i"'] \arrow[u, phantom, "B"] &
\end{tikzcd}
$$

then <u>A commutes</u> $\iff$ <u>B commutes</u>.

Follows from filling Λ^3_1, Λ^3_2.

Or can draw as:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend left, "h"] \arrow[drr, bend right, "k"'] & b \arrow[r, "g"] \arrow[dr, "j"] \arrow[d, phantom, "A"] & c \arrow[d, "i"] \\ & {} & d \end{tikzcd}$$

& outside as B.

- Then

$$\begin{tikzcd} a \arrow[r, "1"] \arrow[rr, bend left, "f"] \arrow[drr, bend right, "g"'] & a \arrow[r, "f"] \arrow[dr, "f"] & b \arrow[d, "1"] \\ & & b \end{tikzcd}$$

shows 1 $f \circ 1 \sim g$ $\iff$ 2 $1 \circ f \sim g$.

$1 \circ f \sim g$

- Likewise 3 $\iff$ 4 (swap f & g)

-

$$\begin{tikzcd} a \arrow[r, "f"'] \arrow[rr, bend left, "f"] \arrow[drr, bend right, "g"'] & b \arrow[r, "1"] \arrow[dr, "1"] & b \arrow[d, "1"] \\ & & b \end{tikzcd}$$

$1 \circ g \sim f \Rightarrow$

$1 \circ f \sim g$

shows 2 $\iff$ 4.

Sim 1 $\iff$ 3.

For the equiv. relation,

- $f \simeq f$ since $\begin{tikzcd} & a \arrow[dr, "f"] \arrow[d, phantom, "\circ"] & \\ a \arrow[ur, "1"] \arrow[rr, "f"'] & & b \end{tikzcd}$.
- If $f \simeq g$ then $f \circ 1 \sim g$ so $g \circ 1 \sim f$ so $g \simeq f$.
- If $f \simeq g$ & $g \simeq h$ then

$$\begin{tikzcd}
 & & b \arrow[dd, "1"] \arrow[dl, phantom, "\circ"] \\
a \arrow[r, "1"'] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "h"'] & a \arrow[ur, "g"'] \arrow[dr, "g"] \arrow[r, phantom, "\circ"] \arrow[d, phantom, "\circ"] & {} \\
 & {} & b
\end{tikzcd}$$

gives $f \simeq h$.

$\square$

Prop Composites are unique up to htpy.

Proof Suppose $\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\circ"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c \end{tikzcd}$ & $\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\circ"] & \\ a \arrow[ur, "f"] \arrow[rr, "i"'] & & c \end{tikzcd}$.

Then

$$\begin{tikzcd}
 & & c \arrow[dd, "1"] \\
a \arrow[r, "f"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h'"'] & b \arrow[ur, "g"] \arrow[dr, "g"'] \arrow[r, phantom, "\circ"] \arrow[u, phantom, "\circ"] \arrow[d, phantom, "\circ"] & {} \\
 & {} & c
\end{tikzcd}$$

shows $h \simeq h'$.

Now we define a cat $\underline{ho(X)}$:

- obs as in X_0
- arrows are htpy classes of arrows.

Composition

- Given $a \xrightarrow{[f]} b \xrightarrow{[g]} c$ define $[g] \circ [f] : a \longrightarrow c$ to be $[h] : a \longrightarrow c$ where $h \sim g \circ f$. Check well-defined.

- Identity $a \xrightarrow{[1_a]} a$.

Associativity

$$
\begin{tikzcd}
 & & c \arrow[dd, "h"] \\
a \arrow[r, "f"] \arrow[urr, bend left, "j"] \arrow[drr, bend right, "k"'] & b \arrow[ur, "g"] \arrow[dr, "i"] & \\
 & & d
\end{tikzcd}
$$

suppose $[g] \circ [f] = [j]$
$[h] \circ [g] = [i]$
$[i] \circ [f] = [k]$

Then $([h] \circ [g]) \circ [f] = [i] \circ [f] = [k]$.

& $[h] \circ ([g] \circ [f]) = [h] \circ [j]$ so must show $[h] \circ [j] = [k]$. See diagram.

Clearly unital & so a category.

Exercise

$ho(X)$ as above equals $h(X)$, as earlier constructed.

L8 - Other models of $(\infty,1)$-category

- As mentioned last week there are the classical Quillen model structures on Top & SSet, & a Quillen equivalence

$$\mathrm{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\underset{\perp}{\rightleftarrows}}} \mathrm{SSet}.$$

- Fibrant spaces = all
  fibrant ssets = Kan complexes := simplicial ∞-groupoids.
- So we obtain

$$\mathrm{Ho}(\mathrm{Top}) \simeq \mathrm{Ho}(\mathrm{Kan})$$ saying

topological spaces $\equiv$ simplicial ∞-groupoids, a form of the homotopy hypothesis, appropriate to simplicial setting.

- What should an $(\infty,1)$-cat be?

  A simple answer:

  a category enriched in ∞-groupoids.

  We can take this to mean topologically or simplicially enriched categories.

  We will take SSet-categories,

which include Kan-enriched cats

- A simplicially enriched cat $\mathcal{C}$ has objects $a, b, c, \ldots$, simplicial sets $\mathcal{C}(a,b)$, comp^n $\mathcal{C}(b,c) \times \mathcal{C}(a,b) \to \mathcal{C}(a,c)$, ids $1 \longrightarrow \mathcal{C}(a,a)$, strictly associative & unital.

here assumed to be small.

- We call the elements of $\mathcal{C}(a,b)_n$ n-morphisms.
- Then for each n, we have a cat $\mathcal{C}_n$ of objects & n-morphisms.
- Moreover the face & degeneracy maps $\cdots \mathcal{C}(a,b)_1 \rightleftarrows \mathcal{C}(a,b)_0$ are then id.on.obs functors.

- In this way we can identify simplicially enriched categories with

(2) Functors $\Delta^{op} \longrightarrow Cat_{i.o.}$ where $Cat_{i.o.}$ consists of small categories & identity on objects functors.

There is a third way we will look at later.

Since $\mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{SSet} \xleftarrow{N} \mathrm{Cat}$ preserve products, they induce functors

$$\mathrm{Top}\text{-}\mathrm{Cat} \xrightarrow{\mathrm{Sing}_*} \mathrm{SSet}\text{-}\mathrm{Cat} \xleftarrow{N_*} \mathrm{Cat}\text{-}\mathrm{Cat}$$

- $\mathrm{Sing}_* C$ has same obs as C, homs $\mathrm{Sing}_* C(A,B) = \mathrm{Sing}(C(A,B))$ & comp^n

$$\mathrm{Sing}(C(B,C)) \times \mathrm{Sing}(C(A,B)) \cong \mathrm{Sing}(C(B,C) \times C(A,B))$$
$$\downarrow \mathrm{Sing}(\circ)$$
$$\mathrm{Sing}(C(A,C))$$

- $\mathrm{Sing}_* C$ always enriched in Kan-complexes, ie. enriched in ∞-groupoids.

$$N_* : 2\text{-Cat} = \text{Cat}\text{-Cat} \hookrightarrow \text{sSet-Cat}$$

identifies 2-Cat as a full subcategory of sSet-Cat containing those sSet-enriched cats $\mathcal{C}$ which are locally nerves of cats -

these are hence locally $(\infty,1)$-cats

(certain $(\infty,2)$-cats - a topic for another day.)

Simplicially-enriched cats vs quasicats

- We would like an adjunction

$$\mathcal{S}\text{-Cat} := \mathrm{SSet}\text{-Cat} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{SSet}$$

which means we should give

$$\Delta \longrightarrow \mathcal{S}\text{-Cat}.$$

- Obvious answer:

$$R : \Delta \hookrightarrow \mathrm{Cat} \overset{D}{\hookrightarrow} 2\text{-Cat} \overset{N_*}{\subseteq} \mathcal{S}\text{-Cat}$$

where D views a cat as loc. discrete 2-cat.

- But then $\mathcal{S}\text{-Cat}(R[n], \mathcal{C}) \cong \mathrm{Cat}([n], \underbrace{U\mathcal{C}}_{\text{underlying cat of } \mathcal{C}})$

- Then N_R is just the composite

$$\mathcal{S}\text{-Cat} \xrightarrow{U} \mathrm{Cat} \xrightarrow{N} \mathrm{SSet}$$

$$\mathcal{C} \longmapsto U\mathcal{C} \text{ where } (U\mathcal{C})(a,b) = \mathcal{C}(a,b)_0.$$

Forgets far too much!

- Need a $\mathbf{\Delta} \longrightarrow S\text{-}Cat$ which will encode more info.

- Consider the adjunction

$$\begin{tikzcd} Cat \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & R\text{-}Graph \arrow[l, shift right, "F"'] \end{tikzcd}$$

cat of reflexive graphs

- FX has morphisms -
  - sequences $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$ where each f_i is non-degenerate,
  - $x \xrightarrow{1_x} x$ the chosen degeneracies.

Composition in FX is by
- concatenation / deletion of identities.

- Unit $\eta_X : X \to UFX$ & counit $\varepsilon_C : FUC \to C$ identity-on-objects.
- Comonad $FU \circlearrowleft Cat$ induces @ $C \in Cat$

$$\Delta^{op} \xrightarrow{Res\,C} Cat$$ its simplicial resolution

$$\cdots\cdots FUFUFUC \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; FUFUC \; \substack{\xrightarrow{\varepsilon_{FUC}}\\ \xleftarrow{F\eta_{UC}}\\ \xrightarrow[FU\varepsilon_C]{}} \; FUC \quad \&$$

all of these maps are id on obs - so this defines a simplicially-enriched category $\mathrm{Res}\, C$.

- So n-arrows of $\mathrm{Res}\,C$ are paths of paths of paths ... in C.

- Obtain $\Delta \hookrightarrow \mathrm{Cat} \xrightarrow{\mathrm{Res}} S\text{-}\mathrm{Cat}$ & this is our functor.

- $\mathrm{Res}([0]) = \{ \cdot \}$.
- $\mathrm{Res}([1]) = \{ 0 \xrightarrow{01} 1 \}$ only one non-degen map.
- $\mathrm{Res}([2])$:

$$\begin{tikzcd} & 1 \arrow[dr, "12"] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur, "01"] \arrow[rr, "02"'] & {} & 2 \end{tikzcd}$$

1-arrow

$$[01, 12] \xrightarrow{[[01,12]]} [02]$$

This induces our adjunction

$$\begin{tikzcd} S\text{-}\mathrm{Cat} \arrow[r, shift right, "H = N_{\mathrm{Res}}"'] \arrow[r, phantom, "\perp"] & S\mathrm{Set} \arrow[l, shift right, "Q"'] \end{tikzcd}$$

homotopy coherent nerve.

Then $HC(2) =$ diagrams $\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\alpha \Downarrow"] & c \end{tikzcd}$ in C

where $\alpha : g.f \Rightarrow h$.

If C is a 2-category, viewed as a simplicially-enriched cat, in fact

$HC = NHom([n], C)$

set of normal lax functors from $[n] \longrightarrow C$.

There are model structures on S-Cat & SSet called the Bergner & Joyal model structures respectively whose fibrant objects are the

- Kan enriched cats
- quasicats

& then

$$\begin{tikzcd} S\text{-}Cat \arrow[r, bend right, "H = N_{Kan}"'] \arrow[r, phantom, "\perp"] & SSet \arrow[l, bend right, "\mathfrak{C}"'] \end{tikzcd}$$

is a Quillen equivalence, giving a sense in which these provide the same model of homotopy theory.

Segal categories

- Recall our perspective on simplicial enriched cats as functors
  $* \quad \Delta^{op} \xrightarrow{X} Cat$ whose components are all i.o.

- These correspond to internal cats in $SSet = (\Delta^{op}, Set)$

$$X1 \times_{X0} X1 \longrightarrow X1 \overset{s}{\underset{t}{\overset{\xleftarrow{i}}{\rightrightarrows}}} X0$$ with $X0$ discrete.

- If $\mathcal{C}$ is simplicially enriched, the corresp. internal cat in SSet looks like

$$\sum_{a,b,c \in obC} \mathcal{C}(a,b) \times \mathcal{C}(b,c) \xrightarrow{\circ} \sum_{a,b \in obC} \mathcal{C}(a,b) \overset{s}{\underset{t}{\overset{\xleftarrow{i}}{\rightrightarrows}}} obC$$

- Evaluating at n, it gives an ordinary cat $\mathcal{C}_n$ - the cat of objects of $\mathcal{C}$ & n-morphisms.

- Now an internal cat in SSet extends !'ly to a functor $\Delta^{op} \xrightarrow{X} SSet$ satisfying the <u>Segal condition</u>.

(Functors $\Delta^{op} \xrightarrow{X} SSet$ will be <u>simplicial spaces</u>)

- So a simplicially-enriched cat C is a simplicial space X such that

① $X0$ is discrete.

② X satisfies the Segal condition.

If X satisfies ① it is called a <u>Segal precategory</u>.

- Given a simplicial space X, we always have

$$\begin{tikzcd}[column sep=small] \boxed{0<1} \arrow[d, "\delta_1^2"] \\ \boxed{0<1<2} \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} & X2 \arrow[dl, "s_2 = \mathrm{Segal}_2"'] \arrow[dr, "X\delta_1^2"] & \\ X1 \times_{X0} X1 & & X1 \end{tikzcd} \quad \in SSet$$

- If the Segal condition holds, then composition is encoded by

$$\begin{tikzcd} & X2 \arrow[dr, "X\delta_1^2"] & \\ X1 \times_{X0} X1 \arrow[ur, "(s_2)^{-1}"] & & X1 \end{tikzcd}$$

In a Segal cat, we weaken composition.

Def^n) A Segal cat is a Segal precategory such that the Segal maps

$$Xn \xrightarrow{\quad s_n \quad} X1 \times_{X0} X1 \times \dots \times_{X0} X1$$

are weak equivalences $\in SSet$.

- This kind of weakening can be considered in other contexts as well,

  eg in Cat:

  then a simplicial category

  $$X : \Delta^{op} \longrightarrow \mathrm{Cat} \text{ sat.}$$

  the Segal cond. is a <u>double cat.</u>

- Asking that the Segal maps

  $$X_n \longrightarrow X_1 \times_{X_0} \cdots \times X_1$$

  are equivalences of cats corresponds to <u>pseudo-double cats.</u>

- Asking that $X(0)$ is discrete forces all squares in the double cat.

  $$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha \Downarrow"] & b \arrow[d, equal] \\ a \arrow[r, "g"'] & b \end{tikzcd}$$

  to have trivial vert. components so it looks globular

  $$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha \Downarrow"] & b \end{tikzcd}$$

  & we get <u>bicategories / 2-categories.</u>

- There is a model structure on
$\mathrm{PreCat} \hookrightarrow [\Delta^{op}, sSet]$, the category of Segal precats, whose fibrant objects are the Reedy-fibrant Segal cats.
This is Quillen equivalent to those earlier described.
I would like to at least say what Reedy-fibrancy means.

- Firstly lets revisit the Segal condition.

- Consider the spine inclusions
$$Sp\Delta^n \hookrightarrow \Delta^n \in (\Delta^{op}, Set)$$

Taking weighted limits of $X : \Delta^{op} \to (\Delta^{op}, Set)$

obtain
$$\begin{array}{ccc} \{\Delta^n, X\} & \longrightarrow & \{Sp\Delta^n, X\} \\ \wr\| & & \wr\| \\ X_n & \xrightarrow{\text{segal}} & X_1 \times_{X_0} X_1 \times_{X_0} \cdots X_1 \end{array}$$
so upper horizontal is Segal map.

② Reedy fibrancy

- For each boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ the induced $\{\Delta^n, X\} \longrightarrow \{\partial\Delta^n, X\}$ is a Kan-fibration.

- Since Mono $=$ Cell$\{\partial\Delta^n \hookrightarrow \Delta^n\}$, Reedy-fib. implies $\{V,X\} \to \{U,X\}$ a Kan-fib $\forall\ U \hookrightarrow V$ mono.

- In partic, the Segal maps are then fibrations & so trivial fibrations.
Then have

$$\begin{tikzcd}
 & X_2 \arrow[dl, two heads, "\simeq"', "\text{Segal}"] \arrow[dr, "X\delta^2_1"] & \\
X_1 \times_{X_0} X_1 & & X_1
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & b \arrow[dr, "g"] & \\
a \arrow[ur, "f"] & & c
\end{tikzcd}
\longleftarrow
\begin{tikzcd}[column sep=small]
 & b \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\
a \arrow[ur, "f"] \arrow[rr, "\exists h"'] & {} & c
\end{tikzcd}
\longrightarrow
a \xrightarrow{h} c$$

- Also considering, $\phi \longrightarrow \Delta^n$, it follows that each X_n is a Kan complex.

Complete Segal spaces

- Here one starts again with simplicial spaces.

$$X : \Delta^{op} \longrightarrow (\Delta^{op}, S).$$

① We keep the condition that the Segal maps are weak equivalences.

② We require Reedy fibrancy.

We drop requirement that $X0$ is discrete but add

③ Completeness

- $J \in Cat$ the free iso $0 \rightleftarrows 1$ & consider $N(J)$, the nerve of the free iso
- Can think of it as "Free equiv in an ∞-cat". & $\Delta^0 \hookrightarrow N(J)$ either inclusion.

(This is a gen. triv cof in Joyal model structure.)

- Completeness means that
$$\{N(J), X\} \longrightarrow \{\Delta^0, X\} = X0$$
is a weak equivalence (equally a t.fib).

- Usually it is formulated as saying that its section (corr to $N(J) \longrightarrow \Delta^0$)
$$X0 \longrightarrow \{N(J), X\}$$
is a weak equivalence.

- The idea is that
$$\{N(J), X\} = \mathrm{hoeq}(X) \hookrightarrow X(1)$$
is the object of homotopy equivalences in X.

- This says that the identities map
$$X(0) \longrightarrow \mathrm{hoeq}(X) \text{ is an equiv.}$$
It is closely connected to univalence (Stenzel)

Fun Fact (Stenzel)

$X : \Delta^{op} \longrightarrow (\Delta^{op}, Set)$ is a complete Segal sp
$$\Longleftrightarrow$$
$$(\Delta^{op}, Set)^{op} \xrightarrow{\{-, X\}} (\Delta^{op}, Set)$$
is right Quillen functor from Joyal model structure to classical Kan-model str.

There is a model str. on $(\Delta^{op}, SSet)$ whose fibrant obs are the complete Segal spaces, & it is Quillen equivalent to the others.

Summary

4 simplicial models of $(\infty,1)$-cat this week & last.

1. Quasicats
2. Simplicially enriched cats
3. Segal cats
4. Complete Segal spaces.

All equivalent, via equivalences of model cats.

Lecture 9 –
A model-independent approach to $(\infty,1)$-categories

- Last time:
  different models of $(\infty,1)$-category:
  ① quasicats
  ② Segal cats
  ③ complete Segal spaces
  ④ simplicially enriched cats

- Would be nice to do "∞-category theory" (ie. adjoints, limits etc") in a way that is independent of which definition we use.

- This is the idea of Riehl & Verity's ∞-cosmoi.

- Only applies to 1,2 & 3 above. Simpl. enriched cats have some problems because of their semi-strict nature (eg. the maps between them are strict) which make it problematic working with them.

- We will approach the notion of an ∞-cosmos gradually.

The 2-category of quasicategories

- Let $\mathrm{QCat} \hookrightarrow \mathrm{SSet}$ denote the full subcat of quasicategories.
- It is also cartesian closed:
  1. since q-cats are the fibrant obs, closed under products & 1
  2. - If B is a quasicat, then $\mathrm{SSet}(A,B)$ a quasicat. In partic, $\mathrm{QCat}(A,B)$ a quasicat.
- Have adjunction $\mathrm{Cat} \underset{N}{\overset{h}{\rightleftarrows}} \mathrm{QCat}$ $(h \dashv N)$

as described in L7:

hX has same objects as X & arrows: homotopy-classes $a \xrightarrow{[f]} b$.

- In fact h preserves finite products (follows from this description or since QCat "exponential ideal" in SSet).

- So get $2\text{-}\mathrm{Cat} \underset{N_*}{\overset{H = h_*}{\rightleftarrows}} \mathrm{QCat}\text{-}\mathrm{Cat}$ $(H \dashv N_*)$

$$\begin{tikzcd} 2\text{-}Cat \arrow[r, shift right, "N_*"'] \arrow[r, phantom, "\perp"] & QCat\text{-}Cat \arrow[l, shift right, "H = h_*"'] \end{tikzcd}$$

- For $\mathcal{C}$ enriched in quasicats, $H\mathcal{C}$ a 2-cat:
  - objects as in $\mathcal{C}$,
  - arrows: the objects of $\mathcal{C}(a,b)$
  - 2-cells: homotopy classes of arrows in $\mathcal{C}(a,b)$.
- In other words, $H\mathcal{C}$ has same underlying cat as $\mathcal{C}$ and homotopy classes of 2-cells.

Now QCat is QCat-enriched, so can form hQCat - the 2-category of quasicats:

- obs, arrows as in QCat (∞-cats, ∞-functors)
- 2-cells homotopy classes of "∞-nat t's".

A 2-categorical approach to quasicats

Def^n) An adjunction / equivalence of quasicats is an adjunction / equiv. in the 2-category $h\mathcal{Q}Cat$.

- In el. terms, this means

$$A \underset{u}{\overset{f}{\rightleftarrows}} B \quad +$$

$$\begin{tikzcd} & B \arrow[dr, "u"] & \\ A \arrow[ur, "f"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=3, "\varepsilon \Downarrow"] & & A \end{tikzcd} \quad \& \quad \begin{tikzcd} B \arrow[rr, bend left, "1"] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta \Downarrow"] & B \\ & A \arrow[ur, "u"'] & \end{tikzcd}$$

sat. triangle equations

$$\begin{tikzcd} f \arrow[r, "f\eta"] \arrow[rr, bend right, "1"'] & fuf \arrow[r, "\varepsilon f"] \arrow[d, phantom, "="] & f \\ & {} & \end{tikzcd} \quad \& \quad \begin{tikzcd} u \arrow[r, "\eta u"] \arrow[rr, bend right, "1"'] & ufu \arrow[r, "u\varepsilon"] \arrow[d, phantom, "="] & u \\ & {} & \end{tikzcd}.$$

- Note: this really means that the eq's hold up to homotopy in $\mathcal{Q}\text{-}Cat(A,A)$, $\mathcal{Q}\text{-}Cat(B,B)$ - since looking at homotopy-classes of 2-cells.

- Surprising thing: this captures correct notion of adjunction between ∞-cats.

Corollary: Adjoints & equivalences can be composed (etc).

Proof) Use usual 2-categorical argument in $h\mathcal{Q}Cat$ - no "∞-arguments" needed.

∞-cosmoi Version 0

- Now it turns out that the categories
  - CSS of complete Segal spaces
  - SeCat of Segal categories

  are naturally simplicially enriched - indeed there are product preserving functors
  $$K : CSS, SeCat \longrightarrow SSet$$
  taking values in quasicats, so can define $CSS(A,B) = QCat(KA, KB)$.
- Since CSS, SeCat are QCat-enriched, can form homotopy 2-cats $H(CSS), H(SeCat)$ & again these capture the correct notions of adjunction & equivalence, analytically defined -

ie. the elementary definitions one uses in the specific context, using things like initial obs in "slice ∞-cats".

∞-cosmoi Version 0 ctd

- So if all of "∞-category theory" we care about are adjunctions & equivalences,

Def V0) An ∞-cosmos $\mathcal{C}$ is a QCat-enriched category.

Will call objects of $\mathcal{C}$ "∞-cats" & prove things about them using the 2-category $H\mathcal{C}$.

But ∞-category theory should also concern structures like limits in ∞-cats, & for these the def^n above is not enough.

- Limits in an ∞-cat A have diagram shape J a simplicial set (not nec. ∞-cat)

- So can't consider $D : J \longrightarrow A$ as a morphism in $\mathcal{Q}Cat$.

- But $[J, A] = SSet(J, A) \in \mathcal{Q}Cat$ is the power of A by J in $\mathcal{Q}Cat$ –

i.e. $\mathcal{Q}Cat(B, [J, \mathbf{A}]) \cong SSet(J, \mathcal{Q}Cat(B, A))$ defining nat iso

a certain kind of weighted limit.

Axiom (Powers)

An ∞-cosmos $\mathcal{C}$ has powers by simplicial sets.

- Then given $A \in \mathcal{C}$, we can form the power $A^J \in \mathcal{C}$
- Taking powers is functorial in SSet – in partic, the map $J \longrightarrow 1 \in SSet$ induces, by the univ. prop. of powers, a diagonal map $\Delta : A \longrightarrow A^J$.

Def^n) A has J-lims if Δ has a right adj
J-colims if Δ has a left adj.

What if we want to capture the colimit of a particular diagram?

- In QCat, can capture a diagram as
$$1 \xrightarrow{X} A^J \qquad \text{as } 1 \text{ is a qcat.}$$

- Then can capture limit of X
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\mathrm{colim}\, X"] \arrow[r, "X"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & A^J \end{tikzcd}$$
as the right adjoint to Δ relative to X.

- What is a relative adjunction in a 2-category?

Firstly, what is a relative adjunction in Cat?

Given a functor $j : A \longrightarrow B$ & $u : C \longrightarrow B$ we call $f : A \longrightarrow C$ left adjoint to u relative to j when there is a nat t.
$$\begin{tikzcd} & C \arrow[d, "u"] \\ A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, shift right=3, "\eta \Uparrow"] & B \end{tikzcd}$$
such that
$$C(fx, y) \longrightarrow B(jx, uy)$$
$$fx \xrightarrow{\alpha} y \longmapsto jx \xrightarrow{\eta x} ufx \xrightarrow{u\alpha} uy$$
a bijection.

(Equivalently, for each $a \in A$, $\exists\, ja \xrightarrow{\eta a} ua'$ u-universal.)

Write $f \dashv_j u$ & say f is j-left adjoint to u.

- In a 2-cat $\mathcal{C}$ we say

$$\begin{tikzcd} & C \arrow[d, "u"] \\ A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, shift right=3, "\eta\Uparrow"] & B \end{tikzcd}$$

exhibits f as j-left adj to u

if $\forall D \in \mathcal{C}$

$$\begin{tikzcd} & \mathcal{C}(D,C) \arrow[d, "u_*"] \\ \mathcal{C}(D,A) \arrow[ur, "f_*"] \arrow[r, "j_*"'] \arrow[ur, phantom, shift right=3, "\eta_*\Uparrow"] & \mathcal{C}(D,B) \end{tikzcd}$$

exhibits $f_* \dashv_{j_*} u_*$ in Cat.

- i.e. define concept representably.

- In elementary terms,

given

$$\begin{tikzcd} D \arrow[r, "y"] \arrow[d, "x"'] \arrow[dr, phantom, "\alpha \Nearrow"] & C \arrow[d, "u"] \\ A \arrow[r, "j"'] & B \end{tikzcd}$$

$\exists !$

$$\begin{tikzcd} D \arrow[r, "y"] \arrow[d, "x"'] \arrow[dr, phantom, "\beta\Uparrow"] & C \\ A \arrow[ur, "f"'] & \end{tikzcd}$$

st

$$\begin{tikzcd} D \arrow[r, "y"] \arrow[d, "x"'] \arrow[dr, phantom, shift left=3, "\beta\Uparrow"] & C \arrow[d, "u"] \\ A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, shift right=3, "\eta\Uparrow"] & B \end{tikzcd} = \alpha.$$

Remarks) • Also j-left adjoints are called absolute left liftings –

left lifting

$$\begin{tikzcd} & C \arrow[d, "u"] \\ A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, shift right=3, "\eta\Uparrow"] & B \end{tikzcd}$$

&

$$\begin{tikzcd} & & C \arrow[d, "u"] \\ D \arrow[r, "g"] & A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, shift right=3, "\eta\Uparrow"] & B \end{tikzcd}$$

left lifting.

- Now we define a relative adjunction $f \dashv_j u$ in a QCat-enriched $\mathcal{C}$ to be a relative adjunction in $h\mathcal{C}$.

- Right adjoints relative to j defined dually - reverse 2-cells.

Pasting Lemma

$$\begin{tikzcd} & A \arrow[d, "u_2"] \\ & B \arrow[d, "u_1"] \\ D \arrow[uur, bend left, "f_2"] \arrow[ur, "f_1"] \arrow[r, "j"'] \arrow[uur, phantom, "\Uparrow \eta_2"] \arrow[ur, phantom, "\Uparrow \eta_1"'] & C \end{tikzcd}$$

Suppose $f_1 \dashv_j u_1$.

Then $f_2 \dashv_{f_1} u_2 \iff f_2 \dashv_j u_2 u_1$.

Proof] Sim. to standard lemma for Kan extensions.

Colimits in an ∞-cat

- One more thing: in an ∞-cosmos $\mathcal{C}$ we don't consider diagrams as morphisms $1 \longrightarrow A^J$.

Eg. in the ∞-cosmos of pted ∞-cats, $\exists$ only one such diagram.
Hence must allow "diagrams" $B \longrightarrow A^J$ for B arbitrary.

Definition) Let $\mathcal{C}$ be an ∞-cosmos.
A colimit of $B \xrightarrow{X} A^J$ is a left adjoint to X relative to Δ:

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ B \arrow[ur, "\mathrm{col}X"] \arrow[r, "X"'] \arrow[ur, phantom, bend right, "\Uparrow"] & A^J \end{tikzcd}.$$

Remark) In QCat, CSS, SegCat suffices to look at diagrams with $B = 1$.

Thm) Left adjoints preserve colims.

Proof) Given
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ C \arrow[ur, "\mathrm{col}X"] \arrow[r, "X"'] \arrow[ur, phantom, "\varepsilon \Uparrow", near end, shift right=3] & A^J \end{tikzcd}$$
$\in \mathcal{C}$ & $F : A \to B$

must prove
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "F"] & B \arrow[d, "\Delta"] \\ C \arrow[ur, "\mathrm{col}X"] \arrow[r, "X"'] \arrow[ur, phantom, "\varepsilon \Uparrow", near end, shift right=3] & A^J \arrow[r, "F^J"'] \arrow[ur, phantom, "="] & B^J \end{tikzcd}$$
exhibits $F\mathrm{col}X$ as rel. left adj to Δ.

Now $(-)^J$ preserves adjunctions, so $F^J \dashv u^J$ & then
$$\begin{tikzcd} & B^J \arrow[d, "u^J"] \\ A^J \arrow[ur, "F^J"] \arrow[r, "1"'] \arrow[ur, phantom, "\eta^J \Uparrow", near end, shift right=3] & A^J \end{tikzcd}$$
rel adj. Then so is
$$\begin{tikzcd} & & B^J \arrow[d, "u^J"] \\ C \arrow[r, "X"] & A^J \arrow[ur, "F^J"] \arrow[r, "1"'] \arrow[ur, phantom, "\eta^J \Uparrow", near end, shift right=3] & A^J \end{tikzcd}$$
.

So by Lemma, suff to show
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "F"] & B \arrow[d, "\Delta"] \\ C \arrow[ur, "\mathrm{col}X"] \arrow[r, "X"'] \arrow[ur, phantom, "\varepsilon \Uparrow", near end, shift right=3] & A^J \arrow[r, "F^J"] \arrow[ur, phantom, "="] \arrow[dr, "1"'] & B^J \arrow[d, "u^J"] \arrow[d, phantom, "\eta^J \Uparrow", shift right=4] \\ & & A^J \end{tikzcd}$$
is abs. left lifting.

But this equals
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "F"] \arrow[dr, "1"'] & B \arrow[d, "u"] \arrow[d, phantom, "\eta \Uparrow", shift right=4] \\ C \arrow[ur, "\mathrm{col}X"] \arrow[r, "1"'] \arrow[ur, phantom, "\varepsilon \Uparrow", near end, shift right=3] & A^J \arrow[dr, "1"'] & A \arrow[d, "\Delta"] \arrow[dl, phantom, "="] \\ & & A^J \end{tikzcd}$$
or
$$\begin{tikzcd} & B \arrow[d, "u"] \\ & A \arrow[d, "\Delta"] \\ C \arrow[uur, bend left, "F\mathrm{col}X"] \arrow[ur, "\mathrm{col}X"] \arrow[r, "1"'] \arrow[ur, phantom, "\varepsilon \Uparrow", near end, shift right=3] \arrow[uur, phantom, "\eta\,\mathrm{col}X \Uparrow", near end, shift right=2] & A^J \end{tikzcd}$$

Further aspects of ∞-cat theory

- For a morphism $F: A \to B$ of ∞-cats, certainly we would like to form comma ∞-cat B/F for instance.
- If $B^{\Delta(1)}$ denotes the ∞-cat of arrows, we have $B^{\Delta(1)} \xrightarrow{cod} B$ induced by restriction along $\Delta(0) \xrightarrow[\delta_0^1]{} \Delta(1)$.
- Now can form

pullback

$$\begin{tikzcd} B/f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B^{\Delta(1)} \arrow[d, "cod"] \\ A \arrow[r, "F"'] & B \end{tikzcd} \qquad \begin{tikzcd} (a, b \xrightarrow{\kappa} fa) \arrow[r, mapsto] \arrow[d, mapsto] & (b \to fa) \arrow[d, mapsto] \\ a \arrow[r, mapsto] & fa \end{tikzcd}$$

Problem) $QCat \hookrightarrow SSet$ not closed under plbs.
However $cod: B^{\Delta 1} \longrightarrow B$ is a fibration & $QCat$ closed under plbs of fibrations.

- In fact $QCat$, CSS & SegCat all arise as fibrant objects in model cats, so come with natural class of maps: the fibrations between fibrant objects.

- In $QCat$, these are the maps with lifting prop against inner horns & $1 \longrightarrow N(J)$, & are called isofibrations.

Complete definition of ∞-cosmos

A QCat enriched cat $\mathcal{C}$ equipped with a class of maps $A \twoheadrightarrow B$ called isofibrations.

These satisfy the following axioms :

Limits

① $\mathcal{C}$ has powers, all small products, pullbacks of isofibrations & limits of countable towers of isofibrations.

Behavior of isofibrations

② $A \twoheadrightarrow 1$ an isofib.

③ If $A \twoheadrightarrow B$ an isofib, then $\mathcal{C}(C,A) \twoheadrightarrow \mathcal{C}(C,B)$ isofib. of qcats.

④ Isofibrations closed under above lim. constructions.

⑤ . If $X \xrightarrow{i} Y$ mono $\in$ sSet, then $A^Y \twoheadrightarrow A^X$ isofib

Moreover, if $A \xrightarrow{f} B$ isofib, then

$$\begin{tikzcd}
A^Y \arrow[dr, two heads] \arrow[drr, bend left, "f^Y"] \arrow[ddr, bend right, "A^j"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B^Y \arrow[d, "B^j"] \\
& A^X \arrow[r, "f^X"'] & B^X
\end{tikzcd}$$

an isofib

How do these axioms help us?

Certainly $\Delta^0 \xrightarrow{\delta_0^1} \Delta^1$ is mono

$\Rightarrow \quad A^{\Delta^1} \overset{\text{cod}}{\twoheadrightarrow} A^{\Delta^0} = A$ is isofib.

Hence pullback exists

$$\begin{tikzcd} B/f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B^{\Delta(1)} \arrow[d, "\text{cod}"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

so we can talk about comma ∞-cats, slices etc, & lots of other things.

Lots more stuff to be figured out

Eg. - Coherent, monoidal ∞-cats?

Lecture 10 - $(\infty,2)$ & higher

- In last lectures, we looked at various flavours of $(\infty,1)$-category.

This time, we will look at a few flavours of $(\infty,2)$ & perhaps (∞,n)-category.

- Roughly 3 flavours :
  - n-fold structures (iterated approaches)
  - Θ_n-based structures (replace Δ by Θ_n where $\Theta_1 = \Delta$)
  - Marked structures (simplicial sets with some special simplices)

next week

The iterated approach

Firstly, recall that a complete Segal space is a simplicial space

$X : \Delta^{op} \longrightarrow (\Delta^{op}, S)$

such that

① It is Reedy fibrant - the maps $\{\Delta^n, X\} \longrightarrow \{\partial\Delta^n, X\}$ are Kan Fibrations

② the Segal maps

$$\begin{array}{ccc} \{\Delta^n, X\} & \longrightarrow & \{Sp\Delta^n, X\} \\ \| & & \| \\ X_n & \longrightarrow & X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \end{array}$$

are weak equivalences.

③ It is complete - the maps $\{N(J), X\} \longrightarrow X_0$ are weak equivalences.

① is a strong form of the statement that each X_n is a Kan-complex - ie a space. Then ① + ② says it is an internal cat, up to htpy in spaces & ③ says it is homotopically well behaved, somehow.

So we can think of complete Segal spaces as:

- homotopical internal cats in spaces.

The generalisation looks like:

$$
\begin{tikzcd}
\text{Categories} \arrow[d, hook] & \equiv & \text{int. cats in Set} \arrow[d, hook] \\
\text{Complete Segal spaces} & \simeq & \text{homotopical int. cats in Spaces}
\end{tikzcd}
$$

- Now 2-cats $\equiv$ double categories (aka 2-fold cats) which are vertically discrete

$$
\begin{tikzcd}
x \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & y
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha \Downarrow"] & y \arrow[d, equal] \\
x \arrow[r, "g"'] & y
\end{tikzcd}
$$

- Now double cats can be seen as functors $\Delta^{op} \longrightarrow Cat$ satisfying Segal cond.

- Or equivalently,

$$\Delta^{op} \times \Delta^{op} \xrightarrow{X} Set$$

such that $\forall n$ the simplicial sets

$$X(n,-), X(-,n) : \Delta^{op} \longrightarrow Set$$

satisfy the Segal condition.

- Vertical discreteness says that $X(0,-) : \Delta^{op} \longrightarrow Set$ is discrete – each map $X(0,0) \longrightarrow X(0,n)$ is identity where $[n] \longrightarrow [0]$ is unique map in Δ.

Hence

Definition

A 2-fold complete Segal space is a 2-fold simplicial space

$$X : \Delta^{op} \times \Delta^{op} \longrightarrow (\Delta^{op}, Set)$$

such that

1. X is Reedy fibrant (to be defined)

2. $\forall n$, the simplicial spaces
$$X(n,-), X(-,n) : \Delta^{op} \longrightarrow SSet$$
are complete Segal spaces.

3. $X(0,-) : \Delta^{op} \longrightarrow SSet$ is homotopically discrete: for $!:[n] \longrightarrow [0] \in \Delta$, the induced map
$$X(0,0) \longrightarrow X(0,n)$$
is a weak equiv of simplicial sets.

Reedy fibrancy in a more general context

A Reedy cat C is a small cat with a degree function $\deg : \mathrm{ob}\, C \longrightarrow \mathbb{N}$ & a strict factorisation system (C_+, C_-) such that

- each non-id map in C_- lowers degree,
- each non-id map in C_+ raises degree.

Examples

(1) Δ has Reedy structure.
- $\deg([n]) = n$
- $\Delta_- =$ surjections, $\Delta_+ =$ injections.

(2) If C, D are Reedy-cats, so is $C \times D$:
- $\deg(c,d) = \deg(c) + \deg(d)$.
- Classes $(C_- \times D_-, C_+ \times D_+)$.

(3) Reflexive globe category $\mathbb{G}_i =$

$$0 \rightleftarrows 1 \rightleftarrows 2 \; \ldots$$ earlier in course.

(4) Also $\mathbb{G}$ – all maps raise degree (inverse cat)

- I think of degree as measuring the complexity of an object.

Boundaries

- If C is a Reedy cat, can form **boundary**
$$\partial C(-,a) \hookrightarrow C(-,a) \quad \in [C^{op}, Set]$$
where $\partial C(b,a) \hookrightarrow C(b,a)$ consists of those maps $b \to a$ factoring through an object of degree lower than a.

Examples

① For Δ, $\partial\Delta(-,n) \hookrightarrow \Delta(-,n)$ is the boundary $\partial\Delta^n \hookrightarrow \Delta^n$ discussed before.

② In $\mathbb{G}$, $\partial\mathbb{G}(-,n)$ is the free parallel pair of $(n-1)$-cells.

③ For $\Delta \times \Delta$, firstly,
$$\Delta\times\Delta(-,(n,m)) = \Delta(-,n) \times \Delta(-,m).$$
$$\partial(\Delta^n \times \Delta^m) =$$
$$\mathrm{im}\,(\partial\Delta^n \times \Delta^m \to \Delta^n \times \Delta^m,\ \Delta^n \times \partial\Delta^m \to \Delta^n \times \Delta^m)$$

Def.) For $X : \mathcal{C}^{op} \to \mathcal{M}$ & $c \in \mathcal{C}$,
$$M_c X = \{\partial\mathcal{C}(-,c), X\},$$ the c'th matching object.

If $\mathcal{M}$ is a model cat, we call X **Reedy-fibrant** if

$$\begin{array}{ccc} X_c & \longrightarrow & M_c X \\ \shortparallel & & \shortparallel \\ \{\mathcal{C}(-,c), X\} & \longrightarrow & \{\partial\mathcal{C}(-,c), X\} \end{array}$$

is a fibration $\forall c \in \mathcal{C}$.

$\boxed{\text{Remark}}$ Reedy-fibrant objects are the fibrant obs in **Reedy model-structure** on $\mathcal{C}$.

Definition completed

A 2-fold complete Segal space is a 2-fold simplicial space

$$X : \Delta^{op} \times \Delta^{op} \longrightarrow (\Delta^{op}, Set)$$

such that

1. X is Reedy fibrant (with $\deg([n],[m]) = n+m$ as above)
2. X is a complete Segal space in each variable.
3. $X(0,-)$ is homotopically discrete.

- Not difficult to extend Θ_n.
- Fibrant obs in model str.
- One can also do Segal n-cats (sim. in. spirit)
- Some refs here:

- Riehl-Verity - Theory & Practice of Reedy cats
- Lyne Moser PhD thesis

Structures based on Θ

- Recall Θ_0 the cat of glob. pasting diags / t.o.d's
- Have

$$\begin{tikzcd} \Theta_0 \arrow[r] \arrow[dr, "\text{id on obs}"'] & {[G^{op}, Set]} \arrow[r, "F"] & \infty\text{-}Cat \\ & \Theta \arrow[ur, tail, "ff"'] & \end{tikzcd}$$

globular theory of strict ω-categories.

- Θ has objects the glob. pasting diags $\bar{n}$ & morphisms

$$F\bar{n} \longrightarrow F\bar{m} \in \infty\text{-}Cat.$$

- $\Theta_n \hookrightarrow \Theta$ is the full subcat of Θ containing the glob. p.d.'s of dimension at most n.
- It is also the n-globular theory of strict n-categories.

  - For instance $\Theta_1 = \Delta$ - obs

    $0, \; 0 \to 1, \; 0 \to 1 \to 2, \; \ldots$ & functors between.

- Θ_2 has objects the 1-d pasting diagrams as above plus the 2-d pasting diagrams like

(2)

$$\begin{tikzcd} 0 \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}$$

$,$

$(2,1,2,0,2)$

$$\begin{tikzcd} 0 \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & 1 \arrow[r, bend left=50, "i"] \arrow[r, bend right=50, "j"'] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd}$$

$.$

- There is a 2-functor

$$F(2) \xrightarrow{\text{comp}} F(2,1,2,0,2)$$

picking out the composite 2-cell

$$\begin{tikzcd} 0 \arrow[r, bend left=40, "if"] \arrow[r, bend right=40, "jh"'] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd}$$

- Θ_n plays same role for (∞,n)-cats as $\Theta_1 = \Delta$ plays for $(\infty,1)$-cats.

n-quasicategories

- These are defined as presheaves
$$\Theta_n^{op} \longrightarrow Set$$
which are fibrant objects in a certain model structure.
- Can be described as injectives, but not so easy or illuminating so I will say no more.

- Will say more about complete Segal Θ_n-spaces (aka higher Rezk spaces) which are functors
$$\Theta_n^{op} \longrightarrow Sp = [\Delta^{op}, Set]$$
- Helpful/fun to have $\rightarrow$

Inductive description of Θ_n

- It is possible to describe the maps in Θ_n using a different encoding of globular pasting diagrams.

Observation: An $(n+1)$-dim g.p.d consists of $[n] \in \Delta$ & for each $i < i+1 \in [n]$ an n-d pasting diagram

Examples)

$$\bullet \quad \begin{tikzcd} 0 \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & 2 \end{tikzcd} \quad \sim \quad 0 \xrightarrow{[1]} 1 \xrightarrow{[2]} 2$$

write as $([2],[1],[2])$

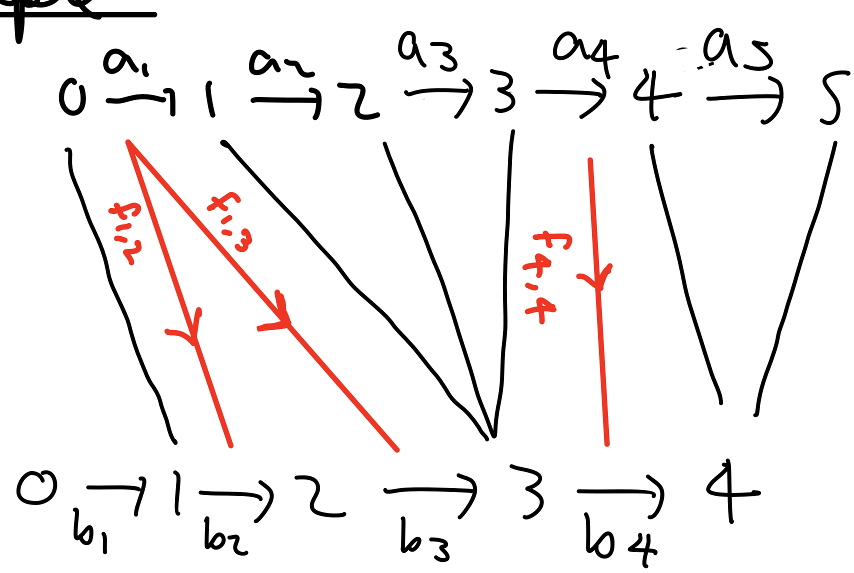
Can capture as iterated wreath product with Δ.

Def^n) Let C be a cat. Define new cat $\underline{\Delta \int C}$ with objects $([n], a)$ where $[n] \in \Delta$ & $a_i \in \mathcal{C}$ for each $1 < i \leq n$.

- A morphism $f : ([n], a) \longrightarrow ([m], b)$ consists of $f : [n] \longrightarrow [m] \in \Delta$ plus

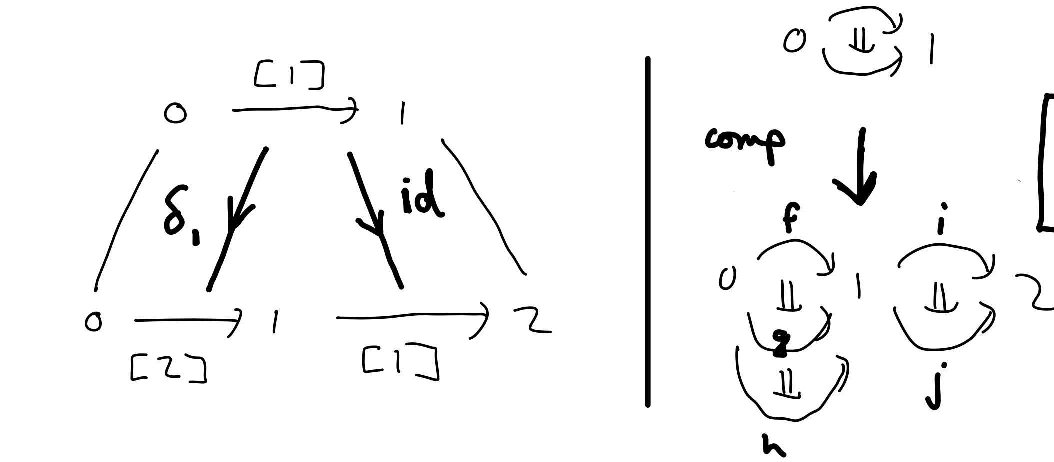
$$a_i \xrightarrow{f_{i,j}} b_j \in \mathcal{C} \text{ whenever } f(i-1) < j < f(i) \text{ for } 1 \leq i \leq n.$$

Example



$\boxed{\text{Theorem}}$ $\Theta_n = \underbrace{\Delta \wr \Delta \wr \Delta \wr \ldots \wr \Delta}_{n \text{ fold}}$

On objects, this is clear.
On arrows, will just give a couple of examples in $\Theta_2 = \Delta \wr \Delta$.



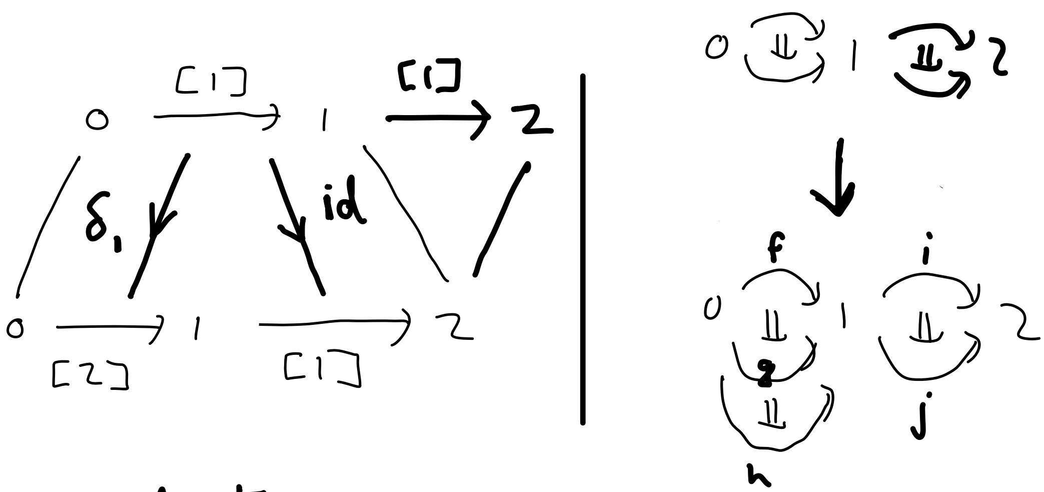
$\boxed{\text{pick out if} \Rightarrow jh.}$

- The interior maps δ_1 & id control the 2-cells we pick out in each hom (vert. composites):

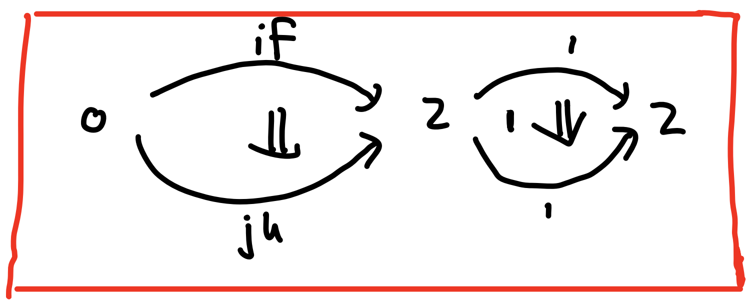
$$\delta_1 \sim \begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}, \quad \mathrm{id} \sim \begin{tikzcd} 1 \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd}$$

- The outer map $\delta_1 : [1] \longrightarrow [2]$ tells us how to put them together (hor. comp)

$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd}.$$



corresponds to



Exercise: prove theorem!

- Outline: a map $(n) \longrightarrow \bar{m}$ in Θ_n

$\sim$ n-cell in $UF(\bar{m})$ - free n-cat.

- Describe this.

- General maps $\bar{n} \longrightarrow \bar{m}$ determined by fact $\bar{n}$ a globular sum.

Fact: - If $\mathcal{C}$ is Reedy, so is $\Delta \wr \mathcal{C}$ -

$\deg([n], a) = n + \sum \deg(a_i)$.

- In partic., Θ_n a Reedy-cat.

E.g.
$$\deg\left(\begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \cdot \end{tikzcd}\right) = 1+1+1 = 3.$$

$$\deg\left(\begin{tikzcd} \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \end{tikzcd}\right) = 2+1+0 = 3.$$

$\boxed{\text{Def}^n}$ A complete Θ_n-Segal space is a functor
$$X : \Theta_n^{op} \longrightarrow [\Delta^{op}, Set]$$
which is

1. Reedy fibrant
2. the Segal maps are weak equivs.
3. satisfies completeness conditions.

- Reedy fibrancy we know!

② The Segal maps

For $\bar{n} = (n_1, n_2, n_3, \dots, n_{2k+1})$ a t.o.d.
this says that the induced

$$X_{\bar{n}} \longrightarrow X_{n_1} \times_{X_{n_2}} X_{n_3} \times \dots \times X_{n_{2k+1}}$$

is a weak equiv. in $[\Delta^{op}, Set]$.

(These maps are invertible just when X is a model of Θ_n - for Set-valued presheaves, this means a strict n-cat. This gives our up to htpy composition)

as Reedy fib implies

$$\begin{tikzcd}
& X_{\bar{n}} \arrow[dl, "\sim", "\text{triv fib}"'] \arrow[dr, "X(\text{comp})"] & \\
X_{n_1} \times_{X_{n_2}} X_{n_3} \times \dots \times X_{n_{2k+1}} \arrow[ur, bend left, "\exists\ \text{section}"] & & X(\dim(\bar{n}))
\end{tikzcd}$$

Remark: Rezk treats "vertical" & "horizontal" composition separately but <u>equiv.</u> to above given Reedy fibrancy.

③ Completeness conditions

- Consider $N : n\text{-Cat} \longrightarrow [\Theta_n^{op}, Set]$ be nerve functor.

- D_m the free n-cat containing an m-cell.
- I_{m+1} the free invertible $(m+1)$-cell.
- $D_m \xrightarrow{d} I_{m+1} \in n\text{-Cat}$ picks out domain of invertible $(m+1)$-cell.

- Eg

	D_m	I_{m+1}
$m=0$	$\bullet$	$0 \rightleftarrows 1$
$m=1$	$0 \to 1$	$\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\Uparrow"] & 1 \end{tikzcd}$

Completeness

$\forall m < n$, the map

$$\begin{array}{ccc} \{NI_{m+1}, X\} & \longrightarrow & \{ND_m, X\} \in [\Delta^{op}, Set] \\ \| & & \cong \\ m\text{-hoeq}(X) & \longrightarrow & X_m \end{array}$$

is a w.e. of simplicial sets.

- When $m=0$, this captures classical completeness condition.

- Next time, marked (∞,n)-cats.

Lecture 11

- I will now turn to the "marked" simplicial examples.
- A few such have been developed. The first such model are the complicial sets, developed by Verity, a model for (∞,∞)-cats.
- Recently Lurie & others have been looking at a similar model for $(\infty,2)$-cats – called ∞-bicategories. These are very similar & motivated by the complicial sets – so we will study the complicial sets.

Idea

- Want non-invertible 2-simplices

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}$$

so that we can capture nerves of 2-cats etc.

- But also need the 2-simplices to encode composition of 1-cells

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\cong"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}$$

& such 2-simplices should be "equivalences".

- Therefore, we need to keep track of a collection of "thin" n-simplices, thought of as equivalences.

- A stratified simplicial set X is a simplicial set with a subset of thin n-simplices containing the degeneracies $\forall n \geq 1$.

  Morphisms of stratified simplicial sets preserve thinness.

- Strat = cat of stratified simp. sets

Def) • $f: X \to Y \in \mathrm{Strat}$ is regular if $a \in X_n$ is thin $\Leftrightarrow$ Fa is thin.

• $f: X \to Y$ is entire if it is the identity on underlying simpl. sets.

Write $F: X \longrightarrow_r Y$ & $F: X \longrightarrow_e Y$ to indicate F is regular/entire.

Note: (Entire, Regular) is fact. system on Strat:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "1"'] & & Y \\ & Z \arrow[ur, "f"'] & \end{tikzcd}$$

where $Z = X$ as simp. sets & $a \in Z$ thin $\Leftrightarrow$ Fa is thin.

Adjunctions:

$$\begin{tikzcd} \mathrm{Strat} \arrow[r, "u" description] & \mathrm{SSet} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

$L \dashv u \dashv R$

where L makes only degeneracies thin & R makes all simplices thin.

Complicial horn inclusions

Defn) Let $0 \leq k \leq n$. Then $\Delta^k[n]$ denotes the n-simplex $\Delta[n]$ & we

- declare a non-degenerate m-simplex to be thin if contains $\{k-1, k, k+1\} \cap [n]$.

Examples - The n-simplex in $\Delta^k[n]$ is thin.

- All $(n-1)$-simps except $(k-1)$'th, k'th & $(k+1)$'th are thin.

- We consider k-horn $\Lambda^k[n] \hookrightarrow_r \Delta^k[n]$ as a stratified simplicial subset by declaring the inclusion to be regular.

Complicial horn inclusions

Def^n) Let $0 \leq k \leq n$. Then $\Delta^k[n]$ denotes the n-simplex $\Delta[n]$ & we

- declare a non-degenerate m-simplex to be thin if contains $\{k-1, k, k+1\} \cap [n]$.

- Below are pictures of the horn inclusions. Those parts in horn are in black, the others in red. Labels are only intended to suggest interpretation.

$\Delta^0(2)$

$$\begin{tikzcd} & 1 \arrow[dr, "fe^{-1}"] & \\ 0 \arrow[ur, "e \simeq"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "s_1"] & & 2 \end{tikzcd}$$

$\Delta^1(2)$

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "gf"'] \arrow[rr, phantom, shift left=3, "s_1"] & & 2 \end{tikzcd}$$

$\Delta^2(2)$

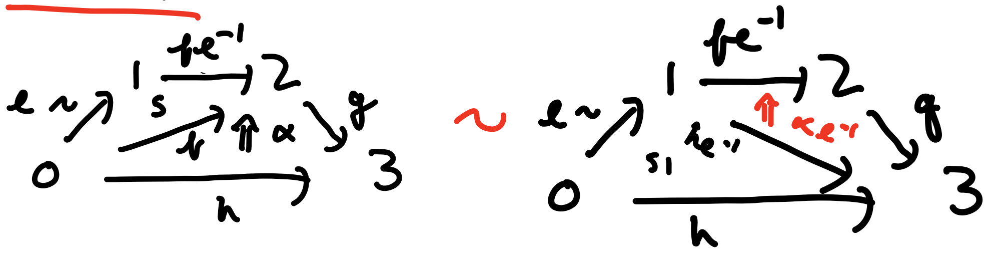
$$\begin{tikzcd} & 1 \arrow[dr, "e \simeq"] & \\ 0 \arrow[ur, "e^{-1}f"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "s_1"] & & 2 \end{tikzcd}$$

$\Delta^0(2) : [-1 < 0 < 1] \cap [0 < 1 < 2] = 0 < 1$

$\Delta^1(2)$ - no thin 1-simps

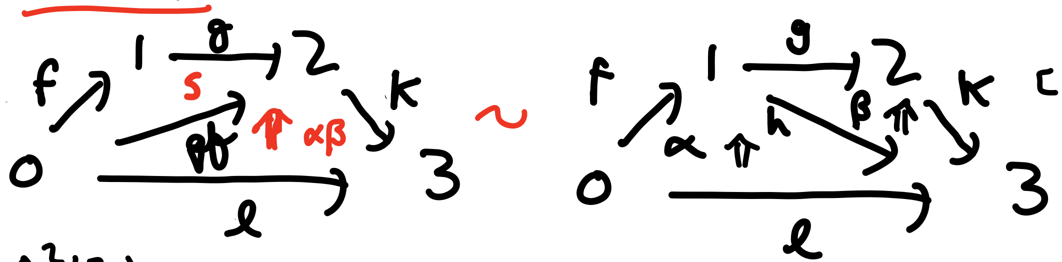
$\Delta^2(2)$ - $1 \to 2$ is thin.

$\Delta^0(3)$



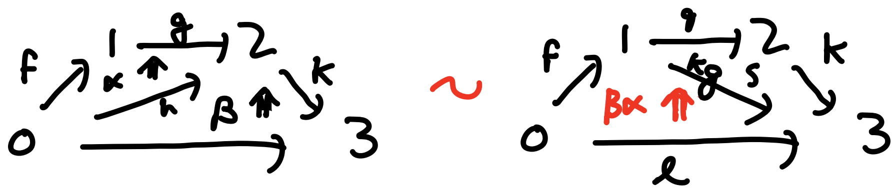
$[-1<0<1]$

$\Delta^1(3)$



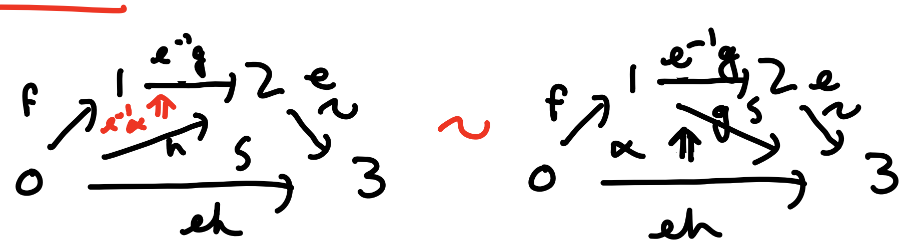
$[0<1<2]$

$\Delta^2(3)$



$[1<2<3]$

$\Delta^3(3)$



$[2<3<4]$

Complicial thinness extensions

- Consider $\Delta^k[n]$.
- Have $\Delta^k[n] \xrightarrow[e]{} \Delta^k[n]' \xhookrightarrow[e]{} \Delta^k[n]''$

  where
  - all have same underlying simplicial set
  - in $\Delta^k[n]'$, declare $(k-1), (k+1)$'th faces thin.
  - in $\Delta^k[n]''$, declare also k'th face thin — so all $(n-1)$-simplices thin.

- Injectivity against $\Delta^k[n]' \xhookrightarrow[e]{} \Delta^k[n]''$ says composite of thin simplices is thin.

- Together, the complicial horn inclusions & complicial thinness extensions are called the elementary (anodyne) extensions.

Def^n) A complicial set is a stratified simplicial set injective wrt to elementary extensions.

(∞,n)-cats

- $X \in \mathrm{Strat}$ is <u>n-trivial</u> if all k-simplices for $k>n$ are thin.

- <u>n-trivial complicial sets</u> provide a model for (∞,n)-cats.

  - Adjunction

$$
\begin{tikzcd}
n\text{-}\mathrm{Strat} \arrow[r, hook] & \mathrm{Strat} \arrow[l, bend right, "\mathrm{tr}_n"'] \arrow[l, bend left, "\mathrm{core}_n"]
\end{tikzcd}
$$

- where tr_n makes all thin for $k>n$,
- core_n restricts to those simplices whose faces above dimension n are all thin.

- The two right adjoints above restrict to complicial sets, giving

$$
(\infty,n)\text{-Cats} \underset{\perp}{\overset{\longrightarrow}{\longleftarrow}} (\infty,\infty)\text{-Cats}
$$

Street-Roberts conjecture

Def^n) A strict complicial set is a strat. simp. set which is orthogonal to the elementary extensions.

There is a nerve functor

$N : \omega\text{-}Cat \longrightarrow Strat$ sending X to its Street nerve, with only identities marked as thin.

Theorem (Verity)

N is fully faithful & has in its essential image exactly the strict complicial sets.

Theorem (Verity)

There is a model structure on Strat whose fibrant objects are the saturated complicial sets: those whose thin simplices are precisely the equivalences.

Remark) Lurie's ∞-bicats are similar, but only considers marked 2-simplices, aka- scaled simplicial sets.

Ref) Emily Riehl: Complicial sets, an overture.