Higher categories course

Plan

Section 1 - Globular stuff

- Globular higher groupoids & cats
- Examples arising from topological spaces & identity types.
- Grothendieck's homotopy hypothesis (precisely)

Section 2 - Simplicial stuff

- Simplicial ∞-groupoids & $(\infty,1)$-cats: Kan complexes & quasicategories
- Different models & the relationship between them: cats enriched in "spaces", Segal categories, complete Segal spaces.
- Some aspects...
- ∞-cosmoi
- Different simplicial models of (∞,n)-cat

& we will see what happens!!

Lecture 1 –
Groupoids & categories revisited

- What are the theories of categories & groupoids & what makes them special (amongst other theories)?
- What do we mean by a "theory"?
- Well, for classical algebraic structures, one can answer this question with Lawvere theories.
- In this setting we are interested in sets X with operations

$$X^{n} \xrightarrow{m_X} X$$

arities are natural numbers

satisfying some equations.

- We want to view operations as maps $1 \xrightarrow[m]{} n$ in a cat $\mathbb{T}$ & our algebra X as a functor

$$\begin{array}{ccc} T^{op} & \xrightarrow{X} & Set \\ m \downarrow & \longmapsto & X^n \\ n & & \downarrow m_X \\ & & X^1 = X \end{array}$$

- For this reason, we take our cat. of arities $\mathbb{N}$ to be the cat of fin. ordinals $n = \{0, \ldots, n-1\}$ for $n \in \mathbb{N}$ & functions between them.

This category has some canonical coproducts

$$\begin{tikzcd} 1 \arrow[dr, "i_0"'] & 1 \arrow[d, "i_1"] \cdots & 1 \arrow[dl, "i_{n-1}"] \\ & n & \end{tikzcd}$$

where i_j picks out the element $j \in 1$.

- A Lawvere theory is an identity on objects

$$J : \mathbb{N} \longrightarrow \mathbb{T}$$

functor preserving these coproducts (equally all finite coproducts)

& a model of $\mathbb{T}$ is a functor

$X : \mathbb{T}^{op} \longrightarrow Set$ sending these coproducts to products

$$\begin{array}{ccc} 1 \searrow & & \nearrow X(1) \\ \vdots \to n & \longmapsto & X(n) \cong X(1)^n \to \vdots \\ 1 \nearrow & & \searrow X(1) \end{array}$$

$\mathrm{Mod}(\mathbb{T}) \subseteq C^{\mathbb{T}}$ is full subcategory of the functor cat. containing the $\mathbb{T}$-models.

Ex.- In the Lawvere theory $\mathbb{T}$ for monoids, we have a map $1 \xrightarrow{m} 2$, which gets sent to $X(1)^2 \cong X(2) \xrightarrow{X(m)} X(1)$ a binary op.

- In general, given a type of algebraic structure $T = (\Omega, \mathbf{E})$, we calculate the associated Lawvere theory $\mathbb{T}$ as follows: consider the adjunction

$$\begin{tikzcd} \mathrm{Alg}(T) \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$$

F - free algebra

U - forgetful

we define $\mathbb{T}$ by factoring

$$\begin{tikzcd} \mathbb{N} \arrow[r, hook, "inc"] \arrow[dr, "I"'] & \mathrm{Set} \arrow[r, "F"] & \mathrm{Alg}(T) \\ & \mathbb{T} \arrow[ur, "J"'] & \end{tikzcd}$$

identity on objects

Fully Faithful

so $\mathbb{T}(n,m) = \mathrm{Alg}(T)(Fn, Fm)$.

with composition as in $\mathrm{Alg}(T)$.

E.g. in the case of monoids
$1 \xrightarrow{m} 2 \in \mathbb{T}$ corresponds to monoid map

$$\begin{array}{ccc} F1 & \longrightarrow & F2 \\ \shortparallel & & \shortparallel \\ \mathbb{N} & \longrightarrow & \mathrm{Words}\{a,b\} \\ 1 & \longmapsto & [a,b] = [a].[b] \end{array}$$

In this ~~setting~~, we always have

$$\begin{tikzcd} \mathrm{Alg}(T) \arrow[rr, "\text{equiv}"] \arrow[dr, "U^T"'] & & \mathrm{Mod}(\mathbb{T}) \arrow[dl, "U^{\mathbb{T}}"] \\ & \mathrm{Set} & \end{tikzcd} \qquad \begin{array}{c} X \\ \downarrow \\ X(1) \end{array}$$

so we can treat classical algebraic structures (involving operations
$$X^n \longrightarrow X \;)$$
using Lawvere Theories.

- But what about categories & groupoids?

A category X is not a set but a directed graph

$$X_1 \overset{s}{\underset{t}{\rightrightarrows}} X_0$$

& involves operations like

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \longrightarrow & X_1 \\ \| & & \| \\ \mathrm{Graph}(\underbrace{0 \to 1 \to 2}_{\text{arities}}, X) & \longrightarrow & \mathrm{Graph}(\underbrace{0 \to 1}_{\text{arities}}, X) \end{array}$$

- As such, we define our category $\Delta_0 \subseteq \mathrm{Graph}$ of arities to consist of the graphs

$$[n] = \{0 \to 1 \to 2 \to \cdots \to n\}$$

for $n \geq 0$. — no endomorphisms!

- In Δ_0, we have the maps

$$[0] \overset{0}{\underset{1}{\rightrightarrows}} [1]$$

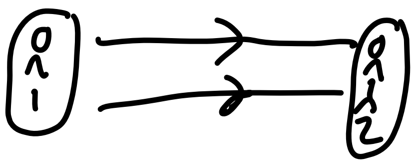
picking out 0 & 1 of $[1]$,

but there are not many maps:
for instance no maps $[2] \to [1]$

$$\begin{array}{ccc} [2] & \longrightarrow & [1] \\ \| & & \| \\ \boxed{0\to1\to2} & \longrightarrow & \boxed{1\to2} \end{array}$$

- The only maps in Δ_0 are the <u>distance preserving embeddings</u> $[n] \to [m]$ for $n < m$.

E.g.



- Each $[n]$ is canonically a <u>wide pushout</u>

$$\begin{tikzcd}
& {[0]} \arrow[dl, "0"'] \arrow[dr, "1"] & & {[0]} \arrow[dl, "0"'] \arrow[dr, "1"] & & & {[0]} \arrow[dl, "0"'] \arrow[dr, "1"] & \\
{[1]} \arrow[drrrr, "i_1"'] & & {[1]} \arrow[drr, "i_2"] & & \cdots & {[1]} \arrow[dl] & & {[1]} \arrow[dlll, "i_n"] \\
& & & & {[n]} & & &
\end{tikzcd}$$

& we'll call <u>these wide pushouts graphical sums</u>.

- A graphical theory consists of an identity on obs functor
$$J : \Delta_0 \longrightarrow \Pi$$
preserving these graphical sums.
- A model of Π is a functor
$$X : \Pi^{op} \longrightarrow Set$$
sending graphical sums in Π to graphical products (wide pullbacks
this just says that the induced map
$$X[n] \longrightarrow \overbrace{X[1] \times_{X[0]} X[1] \times_{X[0]} \cdots \times_{X[0]} X[1]}^{n \text{ copies}}$$
is invertible &
is called the Segal condition.
- $Mod(\Pi) \subseteq [\Pi^{op}, Set]$ is full subcat of presheaves sat. Segal condition.

The theory of categories

We calculate the theory Π_{cat} of categories by factoring

$$\begin{tikzcd}
\Delta_0 \arrow[r, hook] \arrow[dr, "I"', "\text{id on obs}"'] & \mathrm{Graph} \arrow[r, "\text{Free } F"] & \mathrm{Cat} \\
& \Pi_{cat} \arrow[ur, "J", "\text{ff.}"'] &
\end{tikzcd}$$

as before.

What does it look like?

Obs : $[n] = \{0 \to 1 \to \dots \to n\}$.

Morphisms: $F[n] \longrightarrow F[m] \in \mathrm{Cat}$?

Well the free cat on $[n]$ is just $[n]$ viewed as an ordinal (or cat) with all composites & ids added.

Thus $\Pi_{cat} = \Delta$, the simplicial category of finite non-empty ordinals & order-preserving maps between them.

& $\Pi_{cat} = \Delta \xhookrightarrow[J]{} \mathrm{Cat}$ the full inclusion.

- $\Delta_0 \xrightarrow{I} \Delta$ is the obvious id. on obs functor & pres. glob. sums.

This is the graphical theory of categories.

For instance, $\begin{bmatrix}0\\\downarrow\\1\end{bmatrix} \rightrightarrows \begin{bmatrix}0\\\downarrow\\1\\\downarrow\\2\end{bmatrix}$ in Δ (not Δ_0)

corresponds to map $X_1 \times_{X_0} X_1 \xrightarrow{\text{comp}} X_1$ in a category X.

The so-called nerve functor

$$N = \mathrm{Cat}(J-,1) : \mathrm{Cat} \longrightarrow [\Delta^{op}, \mathrm{Set}]$$

sends

$$C \longmapsto NC \text{ where}$$

$$NC(n) = \mathrm{Cat}([n], C) =$$

$\{$ composable sequences $a_0 \xrightarrow{f_0} a_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n$ in C $\}$

It is fully faithful & has in its ess. image those simplicial sets sat. the Segal condition.

this is Grothendieck's nerve theorem.

This says that

$$\mathrm{Cat} \xrightarrow{\simeq} \mathrm{Mod}(\Delta) \hookrightarrow [\Delta^{op}, \mathrm{Set}]$$

so categories $\equiv$ models of Δ.

So indeed we can capture categories using graphical theories.

What about groupoids?

Factoring
$$\begin{tikzcd} \Delta_0 \arrow[r, hook] \arrow[rd, "I"'] & \mathrm{Grph} \arrow[r, "F"] & \mathrm{Gpd} \\ & \Pi_{\mathrm{Gpd}} \arrow[ru, "J"'] & \end{tikzcd}$$

as before, what is a map
$$F[n] \longrightarrow F[m] \in \mathrm{Gpd}\,?$$

Well $F[n] = F\{0 \to 1 \to 2 \to \cdots \to n\}$

$= \{0 \rightleftarrows 1 \rightleftarrows 2 \cdots n-1 \rightleftarrows n\}$ is in fact contractible: non-empty & $\exists!$ iso between any 2 obs.

Because of this, a functor
$$F[n] \longrightarrow F[m]$$ is uniquely specified by the function between sets of objects – i.e. a function $[n] \longrightarrow [m]$ (not nec. ord pres.)

So $\Delta_0 \longrightarrow \Pi_{\mathrm{Gpd}} = \mathbb{F}$

finite non-empty ordinals & functions.

– For instance $\left[\begin{smallmatrix}0\\ \downarrow \\ 1\end{smallmatrix}\right] \rightrightarrows \left[\begin{smallmatrix}1\\ \downarrow \\ 0\end{smallmatrix}\right]$ encodes

$$\begin{tikzcd} X_1 \arrow[rr, "inv"] \arrow[dr, shift right, "s"'] \arrow[dr, shift left, "t"] & & X_1 \arrow[dl, shift right, "t"'] \arrow[dl, shift left, "s"] \\ & X_0 & \end{tikzcd}$$ in a groupoid.

- The inclusion $\mathbb{F} \xhookrightarrow{J} \mathrm{Gpd}$ induces the symmetric nerve functor
$$\mathrm{Gpd} \longrightarrow [\mathbb{F}^{op}, \mathrm{Set}]$$
which restricts to an equivalence $\mathrm{Gpd} \xhookrightarrow{\sim} \mathrm{Mod}(\mathbb{F})$ with those presheaves satisfying the Segal condition.
(This is the symmetric nerve theorem.)

What makes the theory of groupoids special?

- Consider $\Delta_0 \xrightarrow{J} \Pi_{Gpd} = \mathbb{F}$.

recalling $\Pi_{Gpd}([n],[m]) = Gpd(F[n], F[m])$.

where $Gpd \underset{u}{\overset{F}{\rightleftarrows}} Gpd$

- Recall that given $x, y \in UF[n]$, $\exists!\ x \to y$.

ie. given

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & uF[n] \\ {[1]} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

or equivalently,

$$\begin{tikzcd} F[0] \arrow[r, shift left, "\bar{f}"] \arrow[r, shift right, "\bar{g}"'] \arrow[d, shift right, "I0"'] \arrow[d, shift left, "I1"] & F[n] \\ F[1] \arrow[ur, "\exists!h"'] & \end{tikzcd}$$

Def^n) A graphical theory $I : \Delta_0 \to \Pi$ is contractible if given

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "I0"'] \arrow[d, shift left, "I1"] & {[n]} \in \Pi \\ {[1]} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Example) By the above, the theory of

groupoids is contractible.

- In fact, any contractible theory $\mathbb{T}$ encodes groupoids:

e.g. have

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "0"] \arrow[r, shift right, "2"'] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & {[2]} \\ {[1]} \arrow[ur, "\exists ! c"'] & \end{tikzcd} \quad \& \quad \begin{tikzcd} {[0]} \arrow[dr, shift left, "0"] \arrow[dr, shift right, "1"'] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & \\ {[1]} \arrow[r, "\exists ! \, inv"'] & {[1]} \end{tikzcd}$$

$$\& \quad \begin{tikzcd} {[0]} \arrow[dr, shift left, "1"] \arrow[dr, shift right] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & \\ {[1]} \arrow[r, "\exists ! i"'] & {[0]} \end{tikzcd}$$

inducing in a $\mathbb{T}$-model X, the str of a groupoid on it's underlying graph

$$X[1] \underset{x_1}{\overset{x_0}{\rightrightarrows}} X[0]$$

with maps

$$X[1] \times_{X[0]} X[1] \cong X[2] \xrightarrow{X[c]} X[1],$$

$$X[0] \xrightarrow{X[i] \text{ - ids. map}} X[1],$$

$$X[1] \xrightarrow{X[inv]} X[1].$$

Uniqueness of the liftings involved in contractibility ensures associativity etc, so we really obtain a groupoid.

In fact we get a functor

$$\begin{tikzcd}
\mathrm{Mod}(\Pi) \arrow[rr, "K"] \arrow[dr] & \arrow[d, phantom, "="] & \mathrm{Mod}(\Pi_{Gpd}) \cong Gpd \arrow[dl] \\
& {[\Delta_0^{op}, Set]} &
\end{tikzcd}$$

commuting with the forgetful functors to $[\Delta_0^{op}, Set]$,

which is induced by a commutative triangle

$$\begin{tikzcd}
& \Delta_0 \arrow[dl, "I"'] \arrow[dr, "I"] & \\
\Pi_{Gpd} \arrow[rr, "J"'] & & \Pi
\end{tikzcd}$$

(such is called a morph. of graphical theories). This commutative triangle is unique.

That is,

Theorem) Π_{Gpd} is the initial contractible graphical theory.

Proof) - To say that Π is contractible is equally to say that

$$[0] \overset{0}{\underset{1}{\rightrightarrows}} [1]$$ is a

coproduct in $\mathbb{T}$.

- Since $[n]$ is a globular sum, it follows that

$$[0] \underset{n \text{ - picks out } n}{\overset{0 \text{ - picks out } 0}{\underset{\vdots}{\longrightarrow}}} [n]$$

is an $(n+1)$-fold coprod. in $\mathbb{T}$.

- By commutativity in

$$\begin{tikzcd} & \Delta_0 \arrow[dl, "I"'] \arrow[dr, "I'"] & \\ \mathbb{F} \arrow[rr, "J"'] & & \mathbb{T} \end{tikzcd}$$

, J is forced to preserve these coproducts,

so given a function $n \xrightarrow{f} m \in \mathbb{F}$ we must define $Jf : Jn \longrightarrow Jm$ to be the unique map s.t.

$Jf \circ Ji = J(f \circ i)$ for $i \in \{0, \ldots, n\}$.

Functoriality is straightforward.

$\square$