# Lecture 10 - $(\infty,2)$ & higher

- In last lectures, we looked at various flavours of $(\infty,1)$-category.

This time, we will look at a few flavours of $(\infty,2)$ & perhaps $(\infty,n)$-category.

- Roughly 3 flavours :
  - n-fold structures (iterated approaches)
  - $\Theta_n$-based structures (replace $\Delta$ by $\Theta_n$ where $\Theta_1 = \Delta$)
  - Marked structures (simplicial sets with some special simplices)

next week

# The iterated approach

Firstly, recall that a complete Segal space is a simplicial space

$X : \Delta^{op} \longrightarrow (\Delta^{op}, S)$

such that

① It is Reedy fibrant - the maps $\{\Delta^n, X\} \longrightarrow \{\partial\Delta^n, X\}$ are Kan fibrations

② the Segal maps

$$\begin{array}{ccc} \{\Delta^n, X\} & \longrightarrow & \{Sp\Delta^n, X\} \\ \| & & \| \\ X_n & \longrightarrow & X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \end{array}$$

are weak equivalences.

③ It is complete - the maps $\{N(J), X\} \longrightarrow X_0$ are weak equivalences.

① is a strong form of the statement that each $X_n$ is a Kan-complex - ie a space.
Then ① + ② says it is an internal cat, up to htpy in spaces & ③ says it is homotopically well behaved, somehow.

So we can think of complete Segal spaces as:

- homotopical internal cats in spaces.

The generalisation looks like:

$$\begin{tikzcd} \text{Categories} \arrow[d, hook] & \equiv & \text{int. cats in Set} \arrow[d, hook] \\ \text{Complete Segal spaces} & \simeq & \text{homotopical int. cats in Spaces} \end{tikzcd}$$

- Now 2-cats $\equiv$ double categories (aka 2-fold cats) which are vertically discrete

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha \Downarrow"] & y \end{tikzcd} \sim \begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha \Downarrow"] & y \arrow[d, equal] \\ x \arrow[r, "g"'] & y \end{tikzcd}$$

- Now double cats can be seen as functors $\Delta^{op} \longrightarrow Cat$ satisfying Segal cond.
- Or equivalently,
$$\Delta^{op} \times \Delta^{op} \xrightarrow{X} Set$$
such that $\forall n$ the simplicial sets $X(n,-), X(-,n) : \Delta^{op} \longrightarrow Set$ satisfy the Segal condition.

- Vertical discreteness says that $X(0,-) : \Delta^{op} \longrightarrow Set$ is discrete - each map $X(0,0) \longrightarrow X(0,n)$ is identity where $[n] \to [0]$ is unique map in $\Delta$.

Hence

## Definition

A 2-fold complete Segal space is a 2-fold simplicial space

$$X : \Delta^{op} \times \Delta^{op} \longrightarrow (\Delta^{op}, \mathrm{Set})$$

such that

① $X$ is Reedy fibrant (to be defined)

② $\forall n$, the simplicial spaces

$$X(n,-),\ X(-,n) : \Delta^{op} \longrightarrow \mathrm{SSet}$$

are complete Segal spaces.

③ $X(0,-) : \Delta^{op} \longrightarrow \mathrm{SSet}$ is homotopically discrete: for $! : [n] \to [0] \in \Delta$, the induced map

$$X(0,0) \longrightarrow X(0,n)$$

is a weak equiv of simplicial sets.

## Reedy fibrancy in a more general context

A Reedy cat $C$ is a small cat with a degree function $\deg : \mathrm{ob}\, C \longrightarrow \mathbb{N}$ & a strict factorisation system $(C_+, C_-)$ such that

- each non-id map in $C_-$ lowers degree,
- each non-id map in $C_+$ raises degree.

### Examples

① $\Delta$ has Reedy structure.
- $\deg([n]) = n$
- $\Delta_- =$ surjections, $\Delta_+ =$ injections.

② If $C, D$ are Reedy-cats, so is $C \times D$:
- $\deg(c,d) = \deg(c) + \deg(d)$.
- Classes $(C_- \times D_-, C_+ \times D_+)$.

③ Reflexive globe category $\mathbb{G}_i$ =

$$0 \rightleftarrows 1 \rightleftarrows 2 \; \ldots$$ earlier in course.

④ Also $\mathbb{G}$ – all maps raise degree (inverse cat)

- I think of degree as measuring the complexity of an object.

# Boundaries

- If $C$ is a Reedy cat, can form boundary
$$\partial C(-,a) \hookrightarrow C(-,a) \quad \in [C^{op}, Set]$$
where $\partial C(b,a) \hookrightarrow C(b,a)$ consists of those maps $b \to a$ factoring through an object of degree lower than $a$.

## Examples

① For $\Delta$, $\partial\Delta(-,n) \hookrightarrow \Delta(-,n)$ is the boundary $\partial\Delta^n \hookrightarrow \Delta^n$ discussed before.

② In $\mathbb{G}$, $\partial\mathbb{G}(-,n)$ is the free parallel pair of $(n-1)$-cells.

③ For $\Delta\times\Delta$, firstly,
$$\Delta\times\Delta(-,(n,m)) = \Delta(-,n)\times\Delta(-,m).$$
$$\partial(\Delta^n\times\Delta^m) =$$
$$\mathrm{im}\,(\partial\Delta^n\times\Delta^m \to \Delta^n\times\Delta^m,\ \Delta^n\times\partial\Delta^m \to \Delta^n\times\Delta^m)$$

Def.) For $X : \mathcal{C}^{op} \to \mathcal{M}$ & $c \in \mathcal{C}$,
$$M_cX = \{\partial\mathcal{C}(-,c), X\},$$
the $c$'th matching object.

If $\mathcal{M}$ is a model cat, we call $X$ **Reedy-fibrant** if

$$\begin{array}{ccc} X_c & \longrightarrow & M_c X \\ \shortparallel & & \shortparallel \\ \{\mathcal{C}(-,c), X\} & \longrightarrow & \{\partial\mathcal{C}(-,c), X\} \end{array}$$

is a fibration $\forall c \in \mathcal{C}$.

$\boxed{\text{Remark}}$ Reedy-fibrant objects are the fibrant obs in **Reedy model-structure** on $\mathcal{C}$.

**Definition completed**

A 2-fold complete Segal space is a 2-fold simplicial space
$$X : \Delta^{op} \times \Delta^{op} \longrightarrow (\Delta^{op}, Set)$$
such that

1. $X$ is Reedy fibrant (with $\deg([n],[m]) = n+m$ as above)
2. $X$ is a complete Segal space in each variable.
3. $X(0,-)$ is homotopically discrete.

- Not difficult to extend $\Theta_n$.
- Fibrant obs in model str.
- One can also do Segal n-cats (sim. in. spirit)

- Some ref here:

• Riehl-Verity - Theory & Practice of Reedy cats
• Lyne Moser PhD thesis

# Structures based on $\Theta$

- Recall $\Theta_0$ the cat of glob. pasting diags / t.o.d's
- Have

$$\begin{tikzcd} \Theta_0 \arrow[r] \arrow[dr, "\text{id on obs}"'] & {[G^{op}, Set]} \arrow[r, "F"] & \infty\text{-Cat} \\ & \Theta \arrow[ur, tail, "ff"'] & \end{tikzcd}$$

globular theory of strict $\omega$-categories.

- $\Theta$ has objects the glob. pasting diags $\bar{n}$ & morphisms

$$F\bar{n} \longrightarrow F\bar{m} \in \infty\text{-Cat}.$$

- $\Theta_n \hookrightarrow \Theta$ is the full subcat of $\Theta$ containing the glob. p.d.'s of dimension at most $n$.
- It is also the $n$-globular theory of strict $n$-categories.

  - For instance $\Theta_1 = \Delta$ - obs

    $0, \; 0 \to 1, \; 0 \to 1 \to 2, \; \ldots$ & functors between.

- $\Theta_2$ has objects the 1-d pasting diagrams as above plus the 2-d pasting diagrams like

$$\begin{array}{ccc} (2) & & (2,1,2,0,2) \\ \boxed{\begin{tikzcd} 0 \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}} & , & \boxed{\begin{tikzcd} 0 \arrow[r, bend left=60, "f"] \arrow[r, "g" near start] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & 1 \arrow[r, bend left=50, "i"] \arrow[r, bend right=50, "j"'] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd}} \end{array} .$$

- There is a 2-functor

$$F(2) \xrightarrow{\text{comp}} F(2,1,2,0,2)$$

picking out the composite 2-cell

$$\begin{tikzcd} 0 \arrow[r, bend left=50, "if"] \arrow[r, bend right=50, "jh"'] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd}$$

- $\Theta_n$ plays same role for $(\infty,n)$-cats as $\Theta_1 = \Delta$ plays for $(\infty,1)$-cats.

# n-quasicategories

- These are defined as presheaves
$$\Theta_n^{op} \longrightarrow Set$$
which are fibrant objects in a certain model structure.

- Can be described as injectives, but not so easy or illuminating so I will say no more.

- Will say more about complete Segal $\Theta_n$-spaces (aka higher Rezk spaces) which are functors
$$\Theta_n^{op} \longrightarrow Sp = [\Delta^{op}, Set]$$

- Helpful/fun to have →

# Inductive description of $\Theta_n$

- It is possible to describe the maps in $\Theta_n$ using a different encoding of globular pasting diagrams.

**Observation**: An $(n+1)$-dim g.p.d consists of $[n] \in \Delta$ & for each $i < i+1 \in [n]$ an n-d pasting diagram

Examples)

- $$\begin{tikzcd} 0 \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & 2 \end{tikzcd} \quad \sim \quad 0 \xrightarrow{[1]} 1 \xrightarrow{[2]} 2$$

write as $([2],[1],[2])$

Can capture as iterated wreath product with $\Delta$.

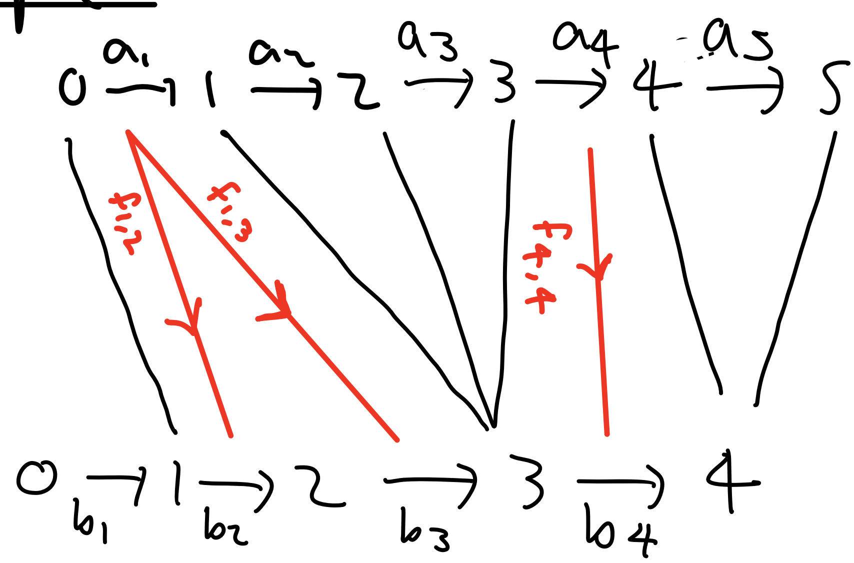
Def$^n$) Let $C$ be a cat. Define new cat $\underline{\Delta S C}$ with objects $([n], a)$ where $[n] \in \Delta$ & $a_i \in \mathcal{C}$ for each $1 < i \leq n$.

- A morphism $f : ([n], a) \longrightarrow ([m], b)$ consists of $f : [n] \longrightarrow [m] \in \Delta$ plus

$$a_i \xrightarrow{f_{i,j}} b_j \in \mathcal{C} \text{ whenever } f(i-1) < j < f(i)$$

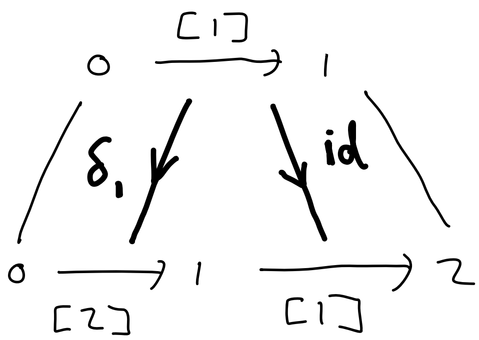
for $1 \leq i \leq n$.

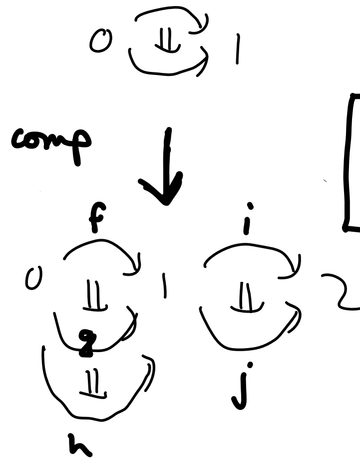
## Example



$\boxed{\text{Theorem}}$ $\Theta_n = \underbrace{\Delta \wr \Delta \wr \Delta \wr \ldots \wr \Delta}_{n \text{ fold}}$

On objects, this is clear.
On arrows, will just give a couple of examples in $\Theta_2 = \Delta \wr \Delta$.



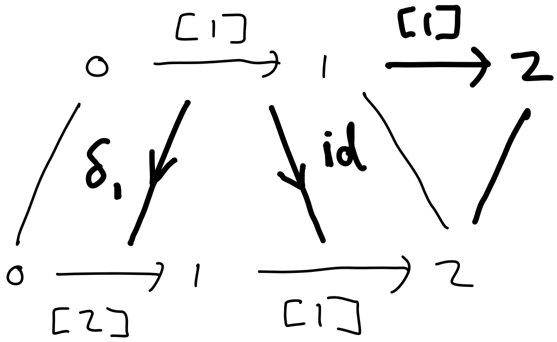

$\boxed{\text{pick out if } \Rightarrow jh .}$

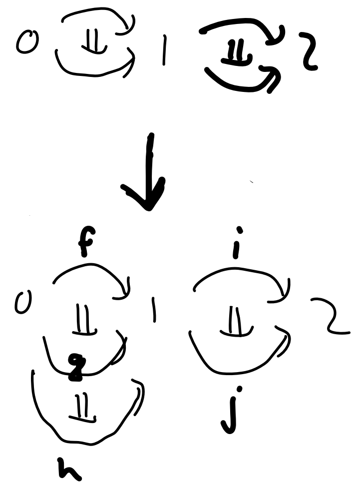
- The interior maps $\delta_1$ & id control the 2-cells we pick out in each hom (vert. composites):

$\delta_1 \sim$
$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}$$
, id $\sim$
$$\begin{tikzcd} 1 \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd}$$

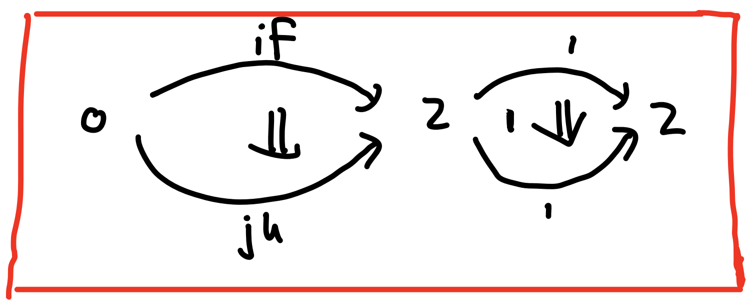
- The outer map $\delta_1 : [1] \longrightarrow [2]$ tells us how to put them together (hor. comp)

$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow"] & 2 \end{tikzcd}.$$




corresponds to




Exercise: prove theorem!

- Outline: a map $(n) \longrightarrow \bar{m}$ in $\Theta_n$

$\sim$ n-cell in $UF(\bar{m})$ - free n-cat.

- Describe this.

- General maps $\bar{n} \longrightarrow \bar{m}$ determined by fact $\bar{n}$ a globular sum.

Fact: - If $\mathcal{C}$ is Reedy, so is $\Delta \wr \mathcal{C}$ -

$$\deg([n], a) = n + \sum \deg(a_i).$$

- In partic., $\Theta_n$ a Reedy-cat.

E.g. $\deg\left(\begin{tikzcd} \cdot \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, bend left=30, "\Downarrow"] \arrow[r, phantom, bend right=30, "\Downarrow"] & \cdot \end{tikzcd}\right) = 1+1+1 = 3.$

$\deg\left(\begin{tikzcd} \cdot \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \cdot \arrow[r] & \cdot \end{tikzcd}\right) = 2+1+0 = 3.$

$\boxed{\text{Def}^n}$ A complete $\Theta_n$-Segal space is a functor

$$X : \Theta_n^{op} \longrightarrow [\Delta^{op}, \text{Set}]$$

which is

1. Reedy fibrant
2. the Segal maps are weak equivs.
3. satisfies completeness conditions.

- Reedy fibrancy we know!

## ② The Segal maps

For $\bar{n} = (n_1, n_2, n_3, \ldots, n_{2k+1})$ a t.o.d. this says that the induced

$$X_{\bar{n}} \longrightarrow X_{n_1} \times_{X_{n_2}} X_{n_3} \times \ldots \times X_{n_{2k+1}}$$

is a weak equiv. in $[\Delta^{op}, \mathrm{Set}]$.

(These maps are invertible just when $X$ is a model of $\Theta_n$ - for Set-valued presheaves, this means a strict n-cat. This gives our up to htpy composition)

$$\begin{tikzcd}
& X_{\bar{n}} \arrow[dl, "\sim"', "\text{triv fib}"] \arrow[dr, "X(\mathrm{comp})"] & \\
X_{n_1} \times_{X_{n_2}} X_{n_3} \times \ldots \times X_{n_{2k+1}} \arrow[ur, bend left, "\exists\text{ section}", "\text{as Reedy fib implies}"'] & & X(\dim(\bar{n}))
\end{tikzcd}$$

Remark : Rezk treats "vertical" & "horizontal" composition separately but equiv. to above given Reedy fibrancy.

## ③ Completeness conditions

- Consider $N : n\text{-Cat} \longrightarrow [\Theta_n^{op}, Set]$ be nerve functor.

- $D_m$ the free n-cat containing an m-cell.
- $I_{m+1}$ the free invertible $(m+1)$-cell.
- $D_m \xrightarrow{d} I_{m+1} \in n\text{-Cat}$ picks out domain of invertible $(m+1)$-cell.
- Eg

| | $D_m$ | $I_{m+1}$ |
|---|---|---|
| $m=0$ | $\bullet$ | $0 \rightleftarrows 1$ |
| $m=1$ | $0 \to 1$ | $0 \rightrightarrows 1$ |

**Completeness**

$\forall m < n$, the map

$$
\begin{array}{ccc}
\{NI_{m+1}, X\} & \longrightarrow & \{ND_m, X\} \in [\Delta^{op}, Set] \\
\shortparallel & & \cong \\
m\text{-hoeq}(X) & \longrightarrow & X_m
\end{array}
$$

is a w.e. of simplicial sets.

- When $m=0$, this captures classical completeness condition.
- Next time, marked $(\infty, n)$-cats.