Cellularity continued

Recap: J a set of arrows in $\mathcal{C}$

Then $J \subseteq \mathrm{Cell}(J) \subseteq \mathrm{Mor}(\mathcal{C})$ consists of closure of J in $\mathrm{Mor}(\mathcal{C})$ under coproducts, pushouts & transfinite composites.

Exercise: $\mathrm{Cell}(J)$ equally consists of the transfinite composites of pushouts of coproducts of maps in J.

(Idea: Just need to show this second class of maps is closed under coproducts, pushouts & transfinite composites.)

Last week: assuming $\mathcal{C}$ locally small, cocomplete, domains of maps in J finitely presentable the (efficient) small object argument produces factorisation

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ \mathrm{Cell}_\omega(J) : \ni & C \arrow[ur, "h"] & \in J^{\square} \end{tikzcd}$$

ω-cellular maps:

the ω-composites of pushouts of coproducts of maps in J.

In fact, under above assumptions, $\mathrm{Cell}_\omega(J) = \mathrm{Cell}(J)$.

Proof of this a bit harder – see Maltsiniotis "Grothendieck ω-groupoids" Proposition A.6

Recall from L3:

$\mathbb{T}$ is a coherator if it is contractible & $\mathbb{T}$ is the colimit of a chain

$$\Theta_0 = \mathbb{T}_0 \to \mathbb{T}_1 \to \cdots \to \mathbb{T}_n \xrightarrow{J_n} \mathbb{T}_{n+1} \to \cdots \to \mathbb{T} \in \mathbb{G}\text{-Th}$$

where - there is a set P_n of parallel pairs $(\ell) \overset{u}{\underset{v}{\rightrightarrows}} \overline{m}$ in $\mathbb{T}_n$ such that $\mathbb{T}_{n+1}$ is obtained by freely adding a lifting

$$\begin{tikzcd} (\ell) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, shift left] \arrow[d, shift right] & \overline{m} \\ (\ell+1) \arrow[ur, "\varphi_{u,v} \in \mathbb{T}_{n+1}"'] & \end{tikzcd}$$

for each $(u,v) \in P_n$.

Now I want to describe a set of maps J in the category of globular theories $\mathbb{G}$-Th such that

or ω-cellular

coherators $\equiv$ J-cellular, J-contractible theories

i.e. $\Theta_0 \longrightarrow \mathbb{T} \longrightarrow 1$

Θ_0 = initial ob

cellular

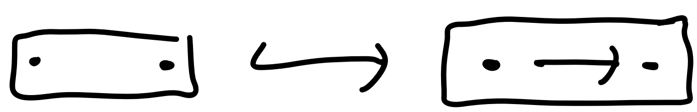
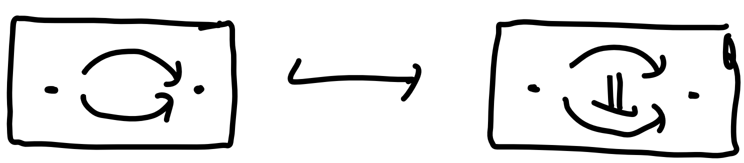
contraction

Firstly, consider the category $[\mathbb{G}^{op}, Set]$

& let $B = \{ \underbrace{\delta Y_n}_{\text{boundary of n-cell}} \xhookrightarrow{j^n} \underbrace{Y_n}_{\text{n-cell}} : n \in \mathbb{N} \}$

defined by $\delta Y_n(m) = \emptyset$ if $m \geq n$

& $\delta Y_n(m) = Y_n(m)$ if $m < n$.

E.g.	$\delta Y_n \hookrightarrow Y_n$
$n=0$	$\emptyset \longrightarrow 1$
$n=1$	
$n=2$	
	- - -

Then δY_n consists of a parallel pair of $(n-1)$-cells, in particular

X is $\mathbb{B}$-injective $\Leftrightarrow$ X is contractible

$$\begin{tikzcd}
\delta Y_n \arrow[r, "\text{parallel } (n-1)\text{-cells } x, y"] \arrow[d] & X \\
Y_n \arrow[ur, dashed, "\exists\, x \to y \in X_n"'] &
\end{tikzcd}$$

Remarks

① Each δY_n, Y_n is finite & so certainly finitely presentable.

② Each globular set X is $\mathbb{B}$-cellular: it is a colimit

$$\emptyset \to X_0 \dashrightarrow X_{\leq n} \hookrightarrow X_{\leq n+1} \cdots \to X$$

where $X_{\leq n}$ has only cells of height $\leq n$ - obtain

$$\begin{tikzcd}
X_{n+1} \cdot \delta Y_{n+1} \arrow[r] \arrow[d] & X_{\leq n} \arrow[d] \\
\underbrace{X_{n+1}}_{\text{set of } (n+1)\text{-cells}} \cdot Y_{n+1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X_{\leq n+1}
\end{tikzcd}$$

i.e. glue on the $(n+1)$-cells @ stage $n+1$.

Consider a globular theory $\mathbb{T}$:

$$\begin{tikzcd} \mathbb{G} \arrow[r, "()"] \arrow[rr, bend right, "D"'] & \Theta_0 \arrow[r, "J"] & \mathbb{T} \end{tikzcd}$$

Recall that $\mathbb{T}$ is contractible,

$\Longleftrightarrow$ each glob. set $\mathbb{T}(D-, \bar{m})$ is contractible

We can view this as a functor

$$\begin{aligned} U_{\bar{m}} : \mathbb{G}\text{-Th} &\longrightarrow [\mathbb{G}^{op}, \mathrm{Set}] \\ \mathbb{T} &\longmapsto \mathbb{T}(D-, \bar{m}) \end{aligned}$$

Let us take for granted

(*) $\mathbb{G}$-Th is cocomplete,

each $U_{\bar{m}}$ has a left adjoint $F_{\bar{m}}$ preserving f.p. objects.

(These props hold much more generally - see eg. Monads & Theories (JB / R. Garner))

Assuming this, we can consider the set of maps in G-Th :

$$B^* = \left\{ F_{\bar{m}}(\partial Y_n) \xrightarrow{F_{\bar{m}}(j_n)} F_{\bar{m}}(Y_n) : \bar{m} \in \Theta_0, n \in \mathbb{N} \right\}$$

all of which have f.p. domain

& now

Theorem

Π is a coherator iff it is B^*-cellular & B^*-contractible.

Proof

- B^*-contractibility of Π says

$$\begin{tikzcd}
F_{\bar{m}}(\partial Y_n) \arrow[r, "\forall f"] \arrow[d, "F_{\bar{m}}(j_n)"'] & \Pi \\
F_{\bar{m}}(Y_n) \arrow[ur, dashed, "\exists g"']
\end{tikzcd}$$

or by adjointness

$$\begin{tikzcd}
\partial Y_n \arrow[r, "\forall f"] \arrow[d, "j_n"'] & U_{\bar{m}}\Pi \\
Y_n \arrow[ur, dashed, "\exists g"']
\end{tikzcd}$$

which says exactly that Π is contractible.

• B^*-cellularity says that $\exists$ colimit

$$\Theta_0 \to \Pi_0 \to \dots \to \Pi_j \to \Pi_{j+1} \to \dots \Pi$$

where $\Pi_j \to \Pi_{j+1}$ is a pushout of a coproduct of maps in B^*.

Let's just consider a pushout of a single map -

in fact to give a commutative square as below left

$$\begin{tikzcd} F_{\bar m}(\partial Y_n) \arrow[r] \arrow[d, "F_{\bar m}(j_n)"'] & \mathbb{R} \arrow[d, "K"] \\ F_{\bar m}(Y_n) \arrow[r] & \$ \end{tikzcd} \qquad \begin{tikzcd} \partial Y_n \arrow[r] \arrow[d, "j_n"'] & \mathbb{R}(D-; \bar m) \arrow[d, "K"] \\ Y_n \arrow[r] & \$(D-, \bar m) \end{tikzcd}$$

is (by adjointness) to give a square as above right -

The top map gives a parallel pair of $(n-1)$-cells $D_n \underset{g}{\overset{f}{\rightrightarrows}} \bar m$ a parallel pair of n-cells in $\bar m$

& the lower map provides a filler

$$\begin{tikzcd} Dn \arrow[r, shift left, "Kf"] \arrow[r, shift right, "Kg"'] \arrow[d, shift right, "KD\sigma"'] \arrow[d, shift left, "KD\tau"] & \overline{m} \\ D(n+1) \arrow[ur, dashed, "\varphi_{f,g}"'] & \end{tikzcd}$$

so the <u>universal such square</u>

$$\begin{tikzcd} F_{\overline{m}}(\partial Y_n) \arrow[r, "\langle f,g \rangle"] \arrow[d, "F_{\overline{m}}(j_n)"'] & \mathbb{R} \arrow[d] \\ F_{\overline{m}}(Y_n) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{R}\langle f,g \rangle \end{tikzcd}$$

has the universal property that it is obtained by <u>freely</u> adding a filler $\varphi_{f,g}$ for the parallel pair (f,g).

More generally, to say $\Pi_j \longrightarrow \Pi_{j+1}$ is a pushout of a coproduct of maps in $\mathcal{B}^*$ is to say that Π_{j+1} is obtained by freely adding fillers for a set of parallel pairs in Π_j. $\square$

Summary

- Contractibility $\sim$ ∞-groupoid str
- Cellularity $\sim$ weakness (no strict equations)

Cellular contractible theories are the theories for weak ∞-groupoids.

These are called coherators.

The homotopy hypothesis made precise

- If $\mathbb{T}$ is a coherator & $\mathbb{S}$ contractible, then $\exists\ \mathbb{T} \xrightarrow{K} \mathbb{S} \in \mathbb{G}\text{-Th}$:

since
$$\begin{tikzcd}
 & \Theta_0 = \emptyset \arrow[dl, "\in \mathcal{B}^*\text{-cell}"'] \arrow[dr] & \\
\mathbb{T} \arrow[rr, dashed, "\exists"] \arrow[dr] & & \mathbb{S} \arrow[dl, "\in \mathcal{B}^*\text{-Inj}"] \\
 & 1 &
\end{tikzcd}$$
&

this induces
$$\begin{tikzcd}
\mathrm{Mod}(\mathbb{S}) \arrow[rr, "K^*"] \arrow[dr, "U^{\mathbb{S}}"'] & \arrow[d, phantom, "="] & \mathrm{Mod}(\mathbb{T}) \arrow[dl, "U^{\mathbb{T}}"] \\
 & {[\mathbb{G}^{op}, \mathrm{Set}]} &
\end{tikzcd}$$
by restr.

Then K^* preserves weak equivalences since it preserves the construction of homotopy groups (exercise) or since, in fact, being a weak equivalence is a property of the underlying map of globular sets.

In particular, if Π is a coherator,

$$\Pi \xrightarrow{\exists K} \Pi_{Top}$$ – globular theory of top. spaces

& so

$$\begin{tikzcd} Top \arrow[r, "N_J"] \arrow[rr, bend left, "N_\infty"] & Mod(\Pi_{Top}) \arrow[r, "K^*"] & Mod(\Pi) \end{tikzcd}$$

preserves weak equivalences since both components do.

Grothendieck's homotopy hypothesis (precise form)

For any coherator Π, the induced functor

$$Top(W^{-1}) \longrightarrow Mod(\Pi)(W^{-1})$$

is an equivalence of categories.

Comments

1. Formulated in 1983 by Grothendieck (Pursuing Stacks)

2. Really one would like a bit more – to put a model structure on $\mathrm{Mod}(\Pi)$ for Π a coherator such that

$$\mathrm{Top} \xrightarrow{N_\infty} \mathrm{Mod}(\Pi)$$

is a Quillen equivalence.

And to prove that for any 2 coherators Π & Π', $\mathrm{Mod}(\Pi)$ & $\mathrm{Mod}(\Pi')$ are suitably equivalent.

Open questions.

③ Smaller question

Is the globular theory Π_{Top} cellular?

ie. is Π_{Top} a coherator?

Examples of coherators

- Both the small object argument & efficient small object argument applied to B^* provide examples of coherators

$$\phi = \Theta_0 \xrightarrow[B^*\text{-cellular}]{} \Pi \xrightarrow[B^*\text{-contractible}]{} 1$$

- The efficient soa produces the free B^*-algebraic injective on ϕ - that is, the initial object of $\mathrm{Alg}(B^*)$ - these are globular theories equipped with a contraction: that is, a theory Π equipped with:

For each parallel pair $n \underset{g}{\overset{f}{\rightrightarrows}} \bar{m} \in \Pi$ we are given a chosen lifting

$$\begin{tikzcd} n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift left] \arrow[d, shift right] & \bar{m} \\ n+1 \arrow[ur, "\varphi(f,g)"'] & \end{tikzcd}$$

This does not follow from the theorem last week on the efficient soa immediately,
since I don't expect
$\mathrm{cell}(B^*) \subseteq \mathrm{Mono}$.

- However it is true that if $\mathbb{T}$ is cellular, then each cellular map
(*) $\mathbb{T} \longrightarrow \mathbb{S}$ is mono (i.e. id on obs & faithful).
- This is all that is needed to show that the efficient soa produces free algebraic injectives on cellular objects
  in particular since
  $\emptyset = \Theta_0$ is cellular,
the efficient soa applied to $\Theta_0 \longrightarrow 1$ produces the initial globular theory with contraction.

(*) above a bit technical to prove - JB "Iterated algebraic injectivity..."