Lecture 6 - Grothendieck ω-groupoids from type theory & weak factorisation systems

This time:
how to get an internal Gr. ω-groupoid str on a topological space, a type, a Kan complex ...

The idea:
If $\mathcal{C}$ has a weak fact. system $(\mathcal{L}, \mathcal{R})$ then can form path object on X:

$$\begin{tikzcd} X \arrow[r, "i \in \mathcal{L}"] \arrow[dr] & PX \arrow[d, "\langle s,t \rangle \in \mathcal{R}"] \\ & X \times X \end{tikzcd},$$

Writing $X_0 = X$, $X_1 = PX$, this is the start of a globular object.

- Now paths are supposed to be composable so we ought to have

$$X_1 \times_{X_0} X_1 \xrightarrow{comp} X_1 .$$

It should be a filler for the

square

$$\begin{tikzcd}
X_0 \arrow[r, "i"] \arrow[d, "\langle i,i \rangle"'] & X_1 \arrow[d, "{\langle s,t \rangle \in \mathcal{R}}"] \\
X_1 \times_{X_0} X_1 \arrow[r, "{\langle s,t \rangle}"'] \arrow[ur, "comp"] & X_0 \times X_0
\end{tikzcd}$$

if we want the composite of id. paths to be an id.

If the left vertical is $\in \mathcal{L}$ - we get such a composition by the lifting prop. & this is the start of an ∞-groupoid str, involving higher paths . . .

Remark: Asking such maps are in $\mathcal{L}$ is closely related to homotopy type theory where if you can define an operation on identity paths you can do it everywhere (path induction)

- This reminds me of something I once knew!
- For topological spaces, the identity path on $x \in X$ is the constant path $\Delta x : [0,1] \longrightarrow X$.
- The composite of constant paths in Top is constant!
- This indicates that the theory Π_{Top} is not cellular:

indeed we have an equation

$$\begin{tikzcd} D1 \arrow[r, "const"] \arrow[d, "comp"'] & D0 \\ D(1,0,1) \arrow[ur, "{\langle const, const \rangle}"'] & \end{tikzcd} \quad \text{in } \Pi_{Top}.$$

- Indeed in Π_{Top}, $D0 = (0)$ is terminal!!

FACT!: If $(0) \in \Pi$ is terminal, Π is not cellular.

Proof: Still to formalise it properly.

Identity type categories

An identity-type category is a cat $\mathcal{C}$ equipped w/ a weak factorisation system $(\mathcal{L}, \mathcal{R})$ such that

- A terminal ob 1 exists & each $! : X \to 1 \in \mathcal{R}$
- Pullbacks of $\mathcal{R}$-maps exist & the pullback of an $\mathcal{L}$-map along an $\mathcal{R}$-map is an $\mathcal{L}$-map.

Remark) • The pullback of an $\mathcal{R}$-map is of course an $\mathcal{R}$-map (true for any wfs).

Examples

Concept introduced by Gambino-Garner (The identity type wfs). Other authors have considered related structures (Joyal - Tribes), Shulman (Type-theoretic fibration cats).

- Main point :- syntactic category of dependent type theory comes equipped w/
  - class of fibrations $\mathcal{R}$.
- Fibrations $B \to A$ correspond to dependent types over the base $x \in A \vdash B(x)$ type.

More accurately, for each dependent type B as above, we have a dependent projection

$$p : (x:A,\ y:B(x)) \longrightarrow (x:A).$$

- If we add "identity types", we get a wfs satisfying the above axioms.

Id types give ① $x, y : A \vdash \mathrm{Id}_A(x,y) : \mathrm{Type}$

& ② $r(x) := \mathrm{refl}(x) \in \mathrm{Id}_A(x,x)$.

- i.e. a factorisation in $\mathbb{S}$ of diagonal.

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[dr, "\Delta"'] & \mathrm{Id}_A \arrow[d, "p"] \\ & A^2 \end{tikzcd}$$

- The rules for identity types say

③ given $x, y : A, z : \mathrm{Id}_A(x,y) \vdash C(x,y,z) \vdash \mathrm{Type}$

& $c(x) : C(x,x,r(x))$

then ④ have $J_c(x,y,u) : C(x,y,u)$

⑤ $J_c(x,x,r(x)) = c(x)$

These correspond to

$$③\ \begin{tikzcd} A \arrow[r, "c"] \arrow[d, "r"'] & C \arrow[d] \\ \mathrm{Id}_A \arrow[r, equal] & \mathrm{Id}_A \end{tikzcd} \quad \text{\& then} \quad \begin{tikzcd} A \arrow[r, "c"] \arrow[d, "r"'] \arrow[dr, phantom, "⑤"] & C \arrow[d] \\ \mathrm{Id}_A \arrow[r, equal] \arrow[ur, "J"', "④"] & \mathrm{Id}_A \end{tikzcd}$$

so the reflexivity maps have left lifting prop wrt dependent projections.

- To make above precise, I point out that the objects of syntactic category are dependent contexts like $(x : A, y : A, z : \mathrm{Id}_A(x, y))$ for instance.

(2) Lots of other examples:
- cat of Kan complexes
- top spaces

Reflexive globular contexts

- Given $X \in \mathcal{C}$ an id. type category, we can extend it to a globular object $X : \mathbb{G}^{op} \to \mathcal{C}$.
- We set $X(0) = X$ & define $X(1)$ by the factorisation

$$\begin{tikzcd}
X(0) \arrow[rr, "i_{0,1} \in \mathcal{L}"] \arrow[drr, "\Delta"'] & & X(1) \arrow[d, "\langle s_1, t_1 \rangle \in \mathcal{R}"] \\
& & X(0) \times X(0)
\end{tikzcd}$$

- This makes $X(1) \overset{s_1}{\underset{t_1}{\rightrightarrows}} X(0)$ a graph or 1-globular object.
- We extend it inductively:

  suppose we have constructed X as an n-globular object, we must show how to extend it to a (n+1)-globular object.

- Given a n-globular object X, can define m-boundary $B_m X$ of X for $1 \leq m \leq n+1$.

- Its universal property is that
$\mathcal{C}(A, B_{m+1}X) \cong$ Parallelpairs in $\mathcal{C}(A, X-)$ of m-cells.

- Therefore
$$\begin{tikzcd}
A \arrow[dr, "\langle f,g\rangle"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& B_{m+1}X \arrow[r, "p_m"] \arrow[d, "q_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "\langle s,t\rangle"] \\
& X_m \arrow[r, "\langle s,t\rangle"'] & (X_{m-1})^2
\end{tikzcd}$$

- This def^n is the clearest one but observe

we have
$$\begin{tikzcd}
X_m \arrow[r, "\exists \langle s,t\rangle"] \arrow[dr, "\langle s,t\rangle"'] & B_m X \arrow[d, "\langle p_{m-1}, q_{m-1}\rangle"] \\
& (X_{m-1})^2
\end{tikzcd}$$

& then
$$\boxed{\begin{tikzcd}
B_{m+1}X \arrow[r, "p_m"] \arrow[d, "q_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "\langle s,t\rangle"] \\
X_m \arrow[r, "\langle s,t\rangle"'] & B_m X
\end{tikzcd}}$$
equivalently since $\langle p_m, q_m\rangle$ is monic.

- The inductive construction now works as: suppose we have the n-glob. object X & suppose that $B_m X$ exists &

$$X_m \xrightarrow[\langle s,t\rangle]{} B_m X \in \mathcal{R} \quad \forall m \leq n.$$

(Note this is true for $n=1$ since

$$X_1 \xrightarrow{\langle s,t\rangle} B_1 X = X_0 \times X_0 \in \mathcal{R}.)$$

- Then the pullback

$$\begin{tikzcd}
B_{m+1}X \arrow[r, "p_m \in \mathcal{R}"] \arrow[d, "\mathcal{R} \ni q_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "\langle s,t\rangle \in \mathcal{R}"] \\
X_m \arrow[r, "\langle s,t\rangle \in \mathcal{R}"'] & B_m X
\end{tikzcd}$$

exists

& we have the induced map

$$\begin{tikzcd}
X(n) \arrow[dr, "\langle 1,1\rangle"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\
& B_{n+1}X \arrow[r, "p_n"] \arrow[d, "q_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & X(n) \arrow[d, "\langle s,t\rangle"] \\
& X(n) \arrow[r, "\langle s,t\rangle"'] & B_n X
\end{tikzcd}$$

& define

$$\begin{tikzcd} X(n) \arrow[r, "i_{n,n+1} \in \mathcal{L}"] \arrow[dr, "\langle 1,1 \rangle"'] & X(n+1) \arrow[d, "\langle s_{n+1}, t_{n+1} \rangle \in \mathcal{R}"] \\ & B_{n+1} X \end{tikzcd}$$

& now $X(n+1) \underset{t_{n+1}}{\overset{s_{n+1}}{\rightrightarrows}} X(n)$ extends X to a (n+1)-globular object.

By induction, we obtain a globular object X satisfying

① $\exists$ $\mathcal{L}$-maps $i_{n,n+1} : X(n) \longrightarrow X(n+1)$ making X a reflexive globular object.

② The maps $X(n+1) \underset{t_n}{\overset{s_n}{\rightrightarrows}} X(n)$ & $\langle s_n, t_n \rangle : X(n+1) \longrightarrow B_{n+1} X$ are $\mathcal{R}$-maps.

A globular object with these props will be called a reflexive globular context.

Theorem

Any reflexive glob. context X ad. the structure of Groth ∞-groupoid.

We will construct it using endomorphism theories.

Endomorphism Theories

- Let $G^{op} \xrightarrow{A} \mathcal{C}$ be a globular object.
- If $\mathcal{C}$ has A-glob. products, can right Kan extend

$$\begin{tikzcd} \Theta_0^{op} \arrow[dr, "A(-)"] & \\ G^{op} \arrow[u, "D^{op}"] \arrow[r, "A"'] \arrow[ur, phantom, "="] & \mathcal{C} \end{tikzcd}$$

$A(-)$ ~ globular product preserving functor.

- Now Factor

$$\begin{tikzcd} \Theta_0^{op} \arrow[r, "J_A^{op}"] \arrow[dr, "A(-)"] & (\mathrm{End}A)^{op} \arrow[d, "K_A"] \\ G^{op} \arrow[u, "D^{op}"] \arrow[r, "A"'] & \mathcal{C} \end{tikzcd}$$

J_A^{op} – id. on obs

K_A – f.f.

- Both J_A^{op} & K_A preserve globular products by construction.
- Then $\Theta_0 \xrightarrow{J_A} \mathrm{End}(A)$ pres glob. sums, so it is a globular theory, the endomorphism theory of A.
- Explicitly, $\mathrm{End}(A)(\bar{n}, \bar{m}) = \mathcal{C}(A(\bar{m}), A(\bar{n}))$
- In particular, $K_A : (\mathrm{End}A)^{op} \longrightarrow \mathcal{C}$, $\bar{n} \longmapsto A(\bar{m})$

equips A with structure of $\mathrm{End}A$-model.

Exercise

There is a natural bijection
$\mathbb{G}\text{-Th}(\mathbb{T}, \text{End}A) \cong (\text{Mod}\,\mathbb{T})_A$ – the set of $\mathbb{T}$-model structures on A.

It is obtained by postcomposing by K_A.

In particular, a good way to put Groth. ∞-groupoid structure on $A : \mathbb{G}^{op} \to \mathcal{C}$ is to show $\text{End}A$ is contractible: then the $\text{End}A$-model structure on A exhibits A as an ω-groupoid.

Now $\text{End}A$ is contractible just when for each $\bar{m} \in \Theta_0$, the globular set
$$\text{End}A(JD-, \bar{m}) \text{ is contractible.}$$
But $\text{End}A(JDn, \bar{m}) = \mathcal{C}(A\bar{m}, An)$ so this just says that

- each $\mathcal{C}(A\bar{m}, A-) : \mathbb{G}^{op} \to \mathbf{Set}$ is a contractible globular set:

In el. terms, given $A\bar{m} \underset{g}{\overset{f}{\rightrightarrows}} An$

st $sf = sg$ & $tf = tg$ (f & g are parallel)

$$\begin{tikzcd}
 & A(n+1) \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
A\bar{m} \arrow[ur, "\exists"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & An
\end{tikzcd}.$$

This is how we will construct an ω-groupoid from type theory.

Theorem

Any reflexive glob. context A ad. the structure of Groth ∞-groupoid.

~~Proof~~ - Let us write $A(n) \xrightarrow{i_{n,m}} A(m)$ for $m > n$ for the maps obtained by composing the $i_{n,n+1}$'s,

& $A(0) \xrightarrow{i_m := i_{0,m}} A(m)$.

- These give a cone $\Delta A(0) \xrightarrow{i} A \in [\mathbb{G}^{op}, \mathcal{C}]$

since

$$\begin{tikzcd}
 & A(n) \arrow[d, shift right, "s_n"'] \arrow[d, shift left, "t_n"] \\
A(0) \arrow[ur, "i_n"] \arrow[r, "i_{n-1}"'] \arrow[ur, phantom, "=", near end, shift right=3] & A(n-1)
\end{tikzcd}.$$

- So we obtain $n \longmapsto A(0) \xrightarrow{i_n} A(n)$

$$\begin{tikzcd}
\mathbb{G}^{op} \arrow[r, "i/A"] \arrow[dr, "A"'] \arrow[dr, phantom, "=", shift left=4] & A(0)/\mathcal{C} \arrow[d, "\mathrm{cod}"] \\
 & \mathcal{C}
\end{tikzcd}
\qquad [\ \longmapsto A(n)$$

- We will prove that

1. $A(0)/\mathcal{C}$ has i/A-glob. products

② $\mathrm{End}(i/A)$ is contractible.

Then the composite

$$\mathrm{End}(i/A) \xrightarrow{K_{i/A}} A(0)/\mathcal{C} \xrightarrow{\mathrm{cod}} \mathcal{C}$$

pres. glob. prods

pres all connected lims

preserves globular products & so will exhibit $\mathrm{End}(i/A)$-model str. on A - ie. ω-groupoid structure.

First, we prove ①.

In constructing globular products in $\mathcal{C}$, we need to construct them using iterated pullbacks, since $\mathcal{C}$ only has some pullbacks:

eg.

$$\begin{tikzcd}
 & & & A(n_1,n_2,n_3,n_4,n_5) \arrow[dll] \arrow[ddr] \arrow[dd, phantom, "\vee"] & \\
 & A(n_1,n_2,n_3) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] & & & \\
A(n_1) \arrow[dr, "t"'] & & A(n_3) \arrow[dl, "s"] \arrow[dr, "t"] & & A(n_5) \arrow[dl, "s"] \\
 & A(n_2) & & A(n_4) &
\end{tikzcd}$$

Use induction over length of tod $\bar{n} = (n_1, \dots, n_k)$.

- We will write $A(\bar{n}) \xrightarrow[p_j^{\bar{n}}]{} A(n_j)$ for the limit projection,

& $A(0) \xrightarrow{i_{\bar{n}}} A(\bar{n})$ for glob. prod in $X(0)/\mathcal{C}$

which satisfies
$$\begin{tikzcd} A(0) \arrow[r, "i_{\bar{n}}"] \arrow[dr, "i_j"'] & A(\bar{n}) \arrow[d, "p_j^{\bar{n}}"] \\ & A(n_j) \end{tikzcd}$$

- Certainly $A(0)/\mathcal{C}$ has glob. products for tod's of length 1.

- For $\bar{n}^+ = (n_1, \dots, n_k, n_{k+1}, n_{k+2})$, the glob prod is the pullback

$$\begin{tikzcd} A(\bar{n}^+) \arrow[rr, "p_{k+2}^{\bar{n}^+}"] \arrow[d, "q"', "\in R"] \arrow[dr, phantom, "\lrcorner"] & & A(n_{k+2}) \arrow[d, "s \in R"] \\ A(\bar{n}) \arrow[r, "p_k^{\bar{n}}"'] & A(n_k) \arrow[r, "t \in R"'] & A(n_{k+1}) \end{tikzcd}$$

which exists since $s \in R$.

- The glob. product in $A(0)/\mathcal{C}$ is then the unique map

$$\begin{tikzcd} A(0) \arrow[d, "i_{\bar{n}^+}"] \arrow[rr, bend left, "i_{n_{k+2}}"] & ! & {} \\ {} & & {} \end{tikzcd}$$

$$\begin{tikzcd}
& A(\bar{n}^+) \arrow[r, "p^{n}_{k+2}"] \arrow[d, "q"', "\in R"] \arrow[dr, phantom, "\lrcorner"] & & A(n_{k+2}) \arrow[d, "s \in R"] \\
\arrow[r, bend right, "i_{\bar{n}}"'] & A(\bar{n}) \arrow[r, "p^{\bar{n}}_{k}"'] & A(n_k) \arrow[r, "t \in R"'] & A(n_{k+1})
\end{tikzcd}$$

to the pullback.

- Our inductive construction also proves: <u>each final proj^n $p^{\bar{n}}_k \in R$.</u>

- Indeed, if $p^{\bar{n}}_k \in R$, so is the lower leg in diagram. Hence so is the upper leg by pullback stability.

- We also will prove by induction that <u>each $i_{\bar{n}} : A(0) \longrightarrow A(\bar{n}) \in L$</u>.

- Since the right leg s is <u>split epi</u>, so is its pullback q - indeed:

$$\begin{tikzcd}
A(\bar{n}) \arrow[r, "p^{\bar{n}}_{k}"] \arrow[d, dashed, "\exists !"] \arrow[dd, bend right=60, "1"'] & A(n_k) \arrow[r, "t"] & A(n_{k+1}) \arrow[d, "i"] \arrow[dd, bend left=60, "1"] \\
A(\bar{n}^+) \arrow[rr, "p^{\bar{n}^+}_{k+2}"] \arrow[d, "q"] \arrow[drr, phantom, "\lrcorner"] & & A(n_{k+2}) \arrow[d, "s"] \\
A(\bar{n}) \arrow[r, "p^{\bar{n}}_{k}"'] & A(n_k) \arrow[r, "t"'] & A(n_{k+1})
\end{tikzcd}$$

Since lower square & outer square are

pullbacks, so is upper. As $i \in \mathcal{L}$, so is $i' \in \mathcal{L}$ as pullbacks of $\mathcal{L}$-maps along $\mathcal{R}$-maps are $\mathcal{L}$-maps.

- Remains to show

$$\begin{tikzcd}
A(0) \arrow[r, "i_{\bar n} \in \mathcal{L}"] \arrow[drr, "i_{\bar n^+}"', "\in \mathcal{L}"] & A(\bar n) \arrow[dr, "i' \in \mathcal{L}"] & \\
& & A(\bar n^+)
\end{tikzcd}$$

since the claim then follows by induction.
- Both give $i_{\bar n}$ when postcomposed by pullback projection q.
- Certainly

$$p^{\bar n^+}_{k+2} \circ i_{\bar n^+} = in_{k+2}.$$

Also

$$p^{\bar n^+}_{k+2} \circ i' \circ i_{\bar n} \overset{\text{def}}{=} i \circ t \circ p^{\bar n}_{k} \circ i_{\bar n}$$

$$\underset{\text{def of } i_{\bar n} \text{ as limit}}{=} i \circ t \circ in_k \underset{\text{cone}}{=} i \circ in_{k+1} \underset{\text{def}}{=} in_{k+2}$$

Hence they agree on postcomp. w' the pullback projections

- It remains to prove contractibility of $\mathrm{End}(i/A)$.

- This amounts to showing that given

$$\begin{tikzcd} A(0) \arrow[d, "i_{\bar n}"'] \arrow[dr, "i_m"] & \\ A(\bar n) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A(m) \end{tikzcd} \qquad \text{then} \qquad \begin{tikzcd} A(\bar n) \arrow[r, "\exists h"] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & A(m+1) \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ & A(m) \end{tikzcd}.$$

s.t. $s \circ f = s \circ g$ &
$t \circ f = t \circ g$

- Get $\langle f, g \rangle : A(\bar n) \longrightarrow B_{m+1}A$ &

then

$$\begin{tikzcd} A(0) \arrow[r, "i_{m+1}"] \arrow[d, "\mathcal{L} \ni i_{\bar m}"'] & A(m+1) \arrow[d, "\langle s,t \rangle \in \mathcal{R}"] \arrow[ddr, bend left, "s"] \arrow[ddr, bend left=20, "t"'] & \\ A(\bar m) \arrow[r, "\langle f,g \rangle"'] \arrow[ur, dashed, "\exists h"] \arrow[drr, bend right, "f"] \arrow[drr, bend right=40, "g"'] & B_{m+1}A \arrow[dr, shift left, "p_m"] \arrow[dr, shift right, "q_m"'] & \\ & & A(m) \end{tikzcd}$$

The square commutes since p_m, q_m are jointly mono.
Hence we obtain a diagonal filler

completing the proof. $\square$

References

- This was proven by Van den Berg & Garner "Types are weak ω-groupoids" using Batanin weak ω-cats.
- I wrote a short expository paper showing how their proof can be done much more briefly if we use Grothendieck ω-groupoids, Note on the construction of globular weak ω-groupoids from types, topological spaces etc. which is what this lecture was based on.