

Lecture 6 - Grothendieck ω -groupoids from type theory & weak factorisation systems

This time :

how to get an internal Gr. ω -groupoid str on a topological space, a type, a Kan complex ...

The idea :

If \mathcal{C} has a weak fact. system (L, R) then can form

path object on X :

$$\begin{array}{ccc} X & \xrightarrow{i \in X} & PX \\ & \searrow & \downarrow \langle s, t \rangle \in R \\ & & X \times X \end{array}$$

Writing $X_0 = X$, $X_1 = PX$, this is the start of a globular object.

- Now paths are supposed to be composable so we ought to have

$$X_1 \times_{X_0} X_1 \xrightarrow{\text{comp}} X_1$$

It should be a filler for the

square

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_1 \\ \langle i, i \rangle \downarrow & \nearrow \text{comp} & \downarrow \langle s, t \rangle \in \mathcal{R} \\ X_1 \times_{X_0} X_1 & \xrightarrow{\langle s, t \rangle} & X_0 \times X_0 \end{array}$$

if we want the composite of id. paths to be an id.

If the left vertical is $\in \mathcal{L}$ - we get such a composition by the lifting prop. & this is the start of an ω -groupoid str, involving higher paths ...

Remark: Asking such maps are in \mathcal{L} is closely related to homotopy type theory where if you can define an operation on identity paths you can do it everywhere (path induction)

- This reminds me of something I once knew!
- For topological spaces, the identity path on $x \in X$ is the constant path $\Delta_x: [0,1] \rightarrow X$.
- The composite of constant paths in Top is constant!
- This indicates that the theory Π_{Top} is not cellular:

indeed we have an equation

$$\begin{array}{ccc}
 D_1 & \xrightarrow{\text{const}} & D_0 \\
 \text{comp} \downarrow & \nearrow & \\
 D(1,0,1) & \langle \text{const}, \text{const} \rangle &
 \end{array}
 \text{ in } \Pi_{\text{Top}}.$$

- Indeed in Π_{Top} , $D_0 = (0)$ is terminal!!

FACT! : IF $(0) \in \Pi$ is terminal,
 Π is not cellular.

Proof: Still to formalise it properly.

Identity type categories

An identity-type category is a cat \mathcal{C} equipped w' a weak factorisation system (L, R) such that

- A terminal ob 1 exists & each $! : X \rightarrow 1 \in R$
- Pullbacks of R -maps exist & the pullback of an L -map along an R -map is an L -map.

Remark) • The pullback of an R -map is of course an R -map (true for any wfs).

Examples

Concept introduced by Gambino-Gorner (The identity type wfs). Other authors have considered related structures (Joyal-Tribes), Shulman (Type-theoretic Fibration cats).

- Main point :- syntactic category of dependent \$
type theory comes equipped w'
 - class of fibrations R .
- Fibrations $B \rightarrow A$ correspond to dependent types over the base $x \in A \vdash B(x)$ type.

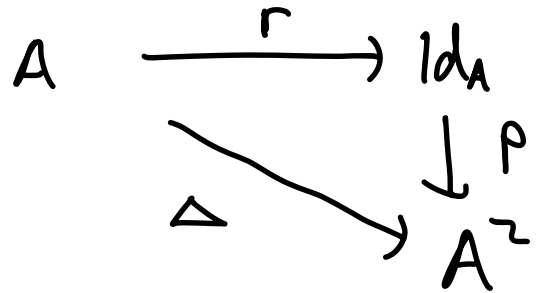
More accurately, for each dependent type B as above, we have a dependent projection

$$p: (x:A, y:B(x)) \longrightarrow (x:A).$$

- If we add "identity types", we get a wfs satisfying the above axioms.

Id types give ① $x, y:A \vdash \text{Id}_A(x, y) : \text{Type}$
 & ② $r(x) := \text{refl}(x) \in \text{Id}_A(x, x)$.

- i.e. a factorisation in \mathcal{S} of diagonal.



- The rules for identity types say

③ given $x, y:A, z:\text{Id}_A(x, y) \vdash C(x, y, z) \vdash \text{Type}$
 & $c(x) : C(x, x, r(x))$

then ④ have $J_c(x, y, u) : C(x, y, u)$

⑤ $J_c(x, x, r(x)) = c(x)$

These correspond to

$$\begin{array}{ccc}
 \textcircled{3} & A & \xrightarrow{c} & C \\
 r \downarrow & & & \downarrow \\
 \text{Id}_A & = & & \text{Id}_A
 \end{array}$$

& then

$$\begin{array}{ccc}
 A & \xrightarrow{c} & C \\
 r \downarrow \textcircled{3} & \nearrow J & \downarrow \\
 \text{Id}_A & = & \text{Id}_A
 \end{array}$$

so the reflexivity maps have left lifting prop wrt dependent projections.

-To make above precise, I point out that the objects of syntactic category are dependent contexts like $(x:A, y:A, z:Id_A(x,y))$ for instance.

- ② Lots of other examples:
- cat of Kan complexes
 - Top spaces

Reflexive globular contexts

- Given $X \in \mathcal{C}$ an id. type category,
we can extend it to a globular
object $X : \mathbb{G}^0 \rightarrow \mathcal{C}$.

- We set $X(0) = X$ & define $X(1)$ by
the factorisation

$$\begin{array}{ccc} X(0) & \xrightarrow{i_0, i_1 \in \mathcal{K}} & X(1) \\ & \searrow \Delta & \downarrow \langle s_1, t_1 \rangle \in \mathcal{R} \\ & & X(0) \times X(0) \end{array}$$

- This makes $X(1) \begin{array}{c} \xrightarrow{s_1} \\ \xrightarrow{t_1} \end{array} X(0)$ a
graph or 1-globular object.

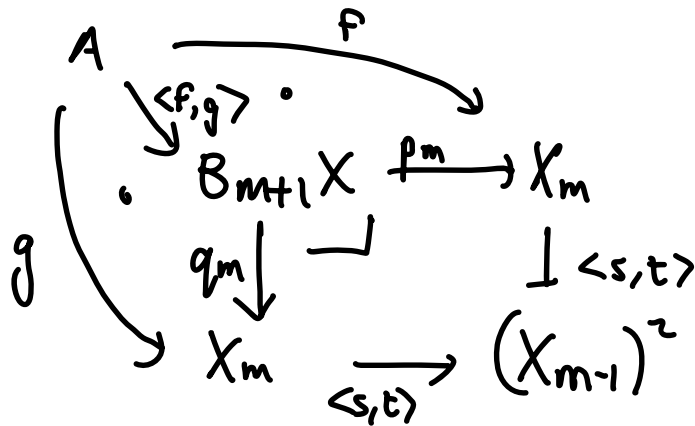
- We extend it inductively:

suppose we have constructed X as an
 n -globular object, we must show how
to extend it to a $(n+1)$ -globular object.

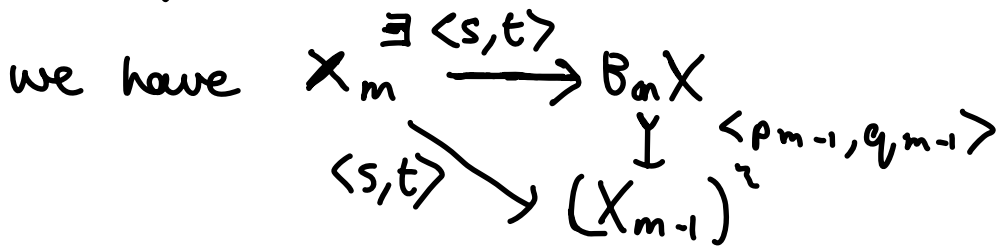
- Given a n -globular object X , can define m -boundary $B_m X$ of X for $1 \leq m \leq n+1$.

- Its universal property is that $\mathcal{C}(A, B_{m+1} X) \cong \text{Parallelopaio in } \mathcal{C}(A, X-)$ of m -cells

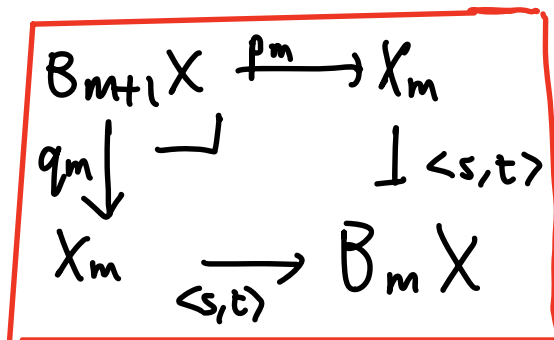
- Therefore



- This defⁿ is the clearest one but observe



& then



equivalently since $\langle p_m, q_m \rangle$ is monic.

- The inductive construction now works as :
suppose we have the n-glob. object X
& suppose that $B_m X$ exists &

$$X_m \xrightarrow{\langle s, t \rangle} B_m X \in \mathcal{R} \quad \forall m \leq n.$$

(Note this is true for $n=1$ since)

$$X_1 \xrightarrow{\langle s, t \rangle} B_1 X = X_0 \times X_0 \in \mathcal{R}.$$

- Then the pullback

$$\begin{array}{ccc} B_{m+1} X & \xrightarrow{p_m \in \mathcal{R}} & X_m \\ \mathcal{R} \ni q_m \downarrow & \lrcorner & \downarrow \langle s, t \rangle \in \mathcal{R} \text{ exists} \\ X_m & \xrightarrow{\langle s, t \rangle \in \mathcal{R}} & B_m X \end{array}$$

& we have the induced map

$$\begin{array}{ccc} X(n) & \xrightarrow{1} & X(n) \\ \downarrow \langle 1, 1 \rangle & & \downarrow \langle s, t \rangle \\ B_{m+1} X & \xrightarrow{p_n} & X(n) \\ \downarrow q_n & \lrcorner & \downarrow \langle s, t \rangle \\ X(n) & \xrightarrow{\langle s, t \rangle} & B_n X \end{array}$$

& define $X(n) \xrightarrow{i_{n,n+1} \in \mathcal{K}} X(n+1)$
 $\searrow \langle 1, 1 \rangle$ $\downarrow \langle s_{n+1}, t_{n+1} \rangle$
 $B_{n+1} X \in \mathcal{R}$

& now $X(n+1) \xrightleftharpoons[t_{n+1}]{s_{n+1}} X(n)$ extends X

to a (n+1)-globular object.

By induction, we obtain a globular object X satisfying

① \exists \mathcal{K} -maps $i_{n,n+1} : X(n) \rightarrow X(n+1)$ making X a reflexive globular object.

② The maps $X(n+1) \xrightleftharpoons[t_n]{s_n} X(n)$ & $\langle s_n, t_n \rangle : X(n+1) \rightarrow B_{n+1} X$ are \mathcal{R} -maps.

A globular object with these props will be called a reflexive globular context.

Theorem

Any reflexive glob. context X ad. the structure of Groth ω -groupoid.

We will construct it using endomorphism theories.

Endomorphism Theories

- Let $G^{\mathcal{P}} \xrightarrow{A} \mathcal{C}$ be a globular object.
- If \mathcal{C} has A -glob. products, we right Kan extend

$$\begin{array}{ccc} \Theta_0^{\mathcal{P}} & \xrightarrow{A(-)} & \mathcal{C} \\ D^{\mathcal{P}} \uparrow & \text{"} & \downarrow \\ G^{\mathcal{P}} & \xrightarrow[A]{} & \mathcal{C} \end{array}$$

globular product preserving functor.

- Now Factor

$$\begin{array}{ccc} \Theta_0^{\mathcal{P}} & \xrightarrow{J_A^{\mathcal{P}}} & (\text{End} A)^{\mathcal{P}} \\ D^{\mathcal{P}} \uparrow & \text{"} & \downarrow K_A \\ G^{\mathcal{P}} & \xrightarrow[A]{} & \mathcal{C} \end{array}$$

id. on obs
f.f.

- Both $J_A^{\mathcal{P}}$ & K_A preserve globular products by construction.

- Then $\Theta_0 \xrightarrow{J_A} \text{End}(A)$ pres glob. sums, so it is a globular theory, the endomorphism theory of A .

- Explicitly, $\text{End}(A)(\bar{n}, \bar{m}) = \mathcal{C}(A(\bar{m}), A(\bar{n}))$

- In particular, $K_A: (\text{End} A)^{\mathcal{P}} \longrightarrow \mathcal{C}$
 $\bar{n} \longmapsto A(\bar{m})$

equips A with structure of $\text{End} A$ -model.

Exercise

There is a natural bijection

$$\mathcal{G}\text{-Th}(\Pi, \text{End}A) \cong (\text{Mod } \Pi)_A - \text{the set of } \Pi\text{-model structures on } A.$$

It is obtained by postcomposing by KA .

In particular, a good way to put Groth.
 ω -groupoid structure on $A: \mathcal{G}^{\text{op}} \rightarrow \mathcal{C}$
is to show $\text{End}A$ is contractible:

Then the $\text{End}A$ -model structure on A
exhibits A as an ω -groupoid.

Now $\text{End}A$ is contractible just when for
each $\bar{m} \in \mathcal{O}_0$, the globular set

$$\text{End}A(\text{JD-}, \bar{m}) \text{ is contractible.}$$

$$\text{But } \text{End}A(\text{JD}_n, \bar{m}) = \mathcal{C}(A\bar{m}, A_n)$$

so this just says that

- each $\mathcal{C}(A\bar{m}, A-): \mathcal{G}^{\text{op}} \rightarrow \text{Set}$
is a contractible globular set:

In el. terms, given $A_{\bar{m}} \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} A_n$
 st $sf = sq$ & $tf = tq$ (f & g are parallel)

$$\begin{array}{ccc} & \exists & A(n+1) \\ & \nearrow & s \perp \downarrow t \\ A_{\bar{m}} & \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{g} \end{array} & A_n \end{array} .$$

This is how we will construct an ω -groupoid from type theory.

Theorem

Any reflexive glob. context A ad. The structure of Groth ω -groupoid.

~~Proof~~ - let us write $A(n) \xrightarrow{i_{n,m}} A(m)$
for $m > n$ for the maps obtained by
composing the $i_{n,n+1}$'s,

& $A(0) \xrightarrow{i_m := i_{0,m}} A(m)$.

- These give a cone $\Delta A(0) \xrightarrow{i} A \in [G^{\mathcal{P}}, \mathcal{C}]$

since

$$A(0) \begin{array}{c} \xrightarrow{i_n} A(n) \\ \xrightarrow{i_{n-1}} A(n-1) \end{array} \begin{array}{c} \text{"} \\ \text{"} \end{array} \begin{array}{c} \downarrow s_n \\ \downarrow t_n \end{array}$$

- So we obtain

$$\begin{array}{ccc} n & \longrightarrow & A(0) \xrightarrow{i_n} A(n) \\ G^{\mathcal{P}} & \xrightarrow{i/A} & A(0)/\mathcal{C} \\ & \searrow & \downarrow \text{cod} \\ & A & \mathcal{C} \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ A(n) \end{array}$$

- We will prove that

(i) $A(0)/\mathcal{C}$ has i/A -glob. products

② $\text{End}(i/A)$ is contractible.

Then the composite

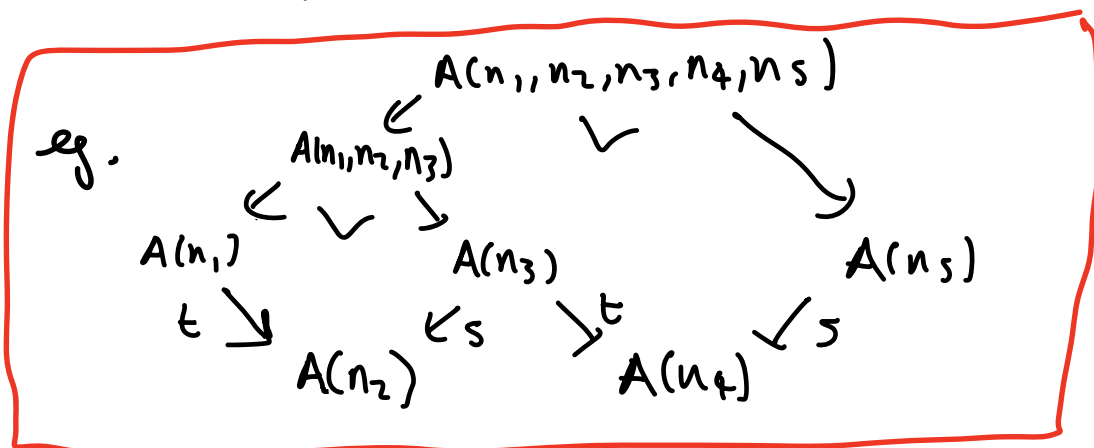
$$\text{End}(i/A) \xrightarrow{K_{i/A}} A(0)/\mathcal{C} \xrightarrow{\text{cod}} \mathcal{C}$$

| pres. glob. prods
└ pres all connected lims

preserves globular products & so will exhibit $\text{End}(i/A)$ -model str. on A - i.e. ω -groupoid structure.

First, we prove ①.

In constructing globular products in \mathcal{C} , we need to construct them using iterated pullbacks, since \mathcal{C} only has some pullbacks:



Use induction over length of $\text{cod } \bar{n} = (n_1, \dots, n_k)$.

- We will write $A(\bar{n}) \xrightarrow{p_j^{\bar{n}}} A(n_j)$ for the limit projection,
 & $A(0) \xrightarrow{i_{\bar{n}}} A(\bar{n})$ for glob. prod in $X(0)/\mathcal{C}$
 which satisfies $A(0) \xrightarrow{i_{\bar{n}}} A(\bar{n})$
 $\searrow i_j \quad \downarrow p_j^{\bar{n}}$
 $A(n_j)$

- Certainly $A(0)/\mathcal{C}$ has glob. products for $\text{cod}'s$ of length 1.

- For $\bar{n}^+ = (n_1, \dots, n_k, n_{k+1}, n_{k+2})$,
 the glob prod is the pullback

$$\begin{array}{ccc} A(\bar{n}^+) & \xrightarrow{p_{k+2}^{\bar{n}^+}} & A(n_{k+2}) \\ \downarrow q \in R & \lrcorner & \downarrow s \in R \\ A(\bar{n}) & \xrightarrow{p_k^{\bar{n}}} A(n_k) \xrightarrow{t \in R} & A(n_{k+1}) \end{array}$$

which exists since $s \in R$.

- The glob. product in $A(0)/\mathcal{C}$ is then the unique map



$$\begin{array}{ccccc}
 & & A(\bar{n}^+) & \xrightarrow{p_{k+2}^{\bar{n}}} & A(n_{k+2}) \\
 & \swarrow i_{\bar{n}} & \downarrow q \in R & \downarrow & \downarrow s \in R \\
 & & A(\bar{n}) & \xrightarrow{p_k^{\bar{n}}} & A(n_k) \xrightarrow{t \in R} & A(n_{k+1})
 \end{array}$$

to the pullback.

- Our inductive construction also proves: each final projⁿ $p_k^{\bar{n}} \in R$.
- Indeed, if $p_k^{\bar{n}} \in R$, so is the lower leg in diagram. Hence so is the upper leg by pullback stability.
- We also will prove by induction that each $i_{\bar{n}} : A(0) \rightarrow A(\bar{n}) \in \mathcal{L}$.
- Since the right leg s is split epi, so is its pullback q .

indeed:

$$\left(\begin{array}{ccccc}
 A(\bar{n}) & \xrightarrow{p_k^{\bar{n}}} & A(n_k) & \xrightarrow{t} & A(n_{k+1}) \\
 \downarrow q & & \downarrow & & \downarrow i \\
 A(\bar{n}^+) & \xrightarrow{p_{k+2}^{\bar{n}}} & A(n_{k+2}) & & \\
 \downarrow q & & \downarrow & & \downarrow s \\
 A(\bar{n}) & \xrightarrow{p_k^{\bar{n}}} & A(n_k) & \xrightarrow{t} & A(n_{k+1})
 \end{array} \right)$$

$\exists ! i$

Since lower square & outer square are

pullbacks, so is upper. As $i \in \mathcal{K}$,
 so is $i \in \mathcal{K}$ as pullbacks
of \mathcal{K} -maps along \mathcal{K} -maps are \mathcal{K} -maps.

- Remains to show

$$\begin{array}{ccc}
 A(0) & \xrightarrow{i_{\bar{n}} \in \mathcal{K}} & A(\bar{n}) & \xrightarrow{i' \in \mathcal{K}} & A(\bar{n}^+) \\
 & \searrow & \xrightarrow{i_{\bar{n}^+} \in \mathcal{K}} & & \\
 & & & & A(\bar{n}^+)
 \end{array}$$

since the claim then follows by induction.

- Both give $i_{\bar{n}}$ when postcomposed
 by pullback projection q .

- Certainly

$$p_{k+2}^{\bar{n}^+} \circ i_{\bar{n}^+} = i_{n_{k+2}}.$$

Also

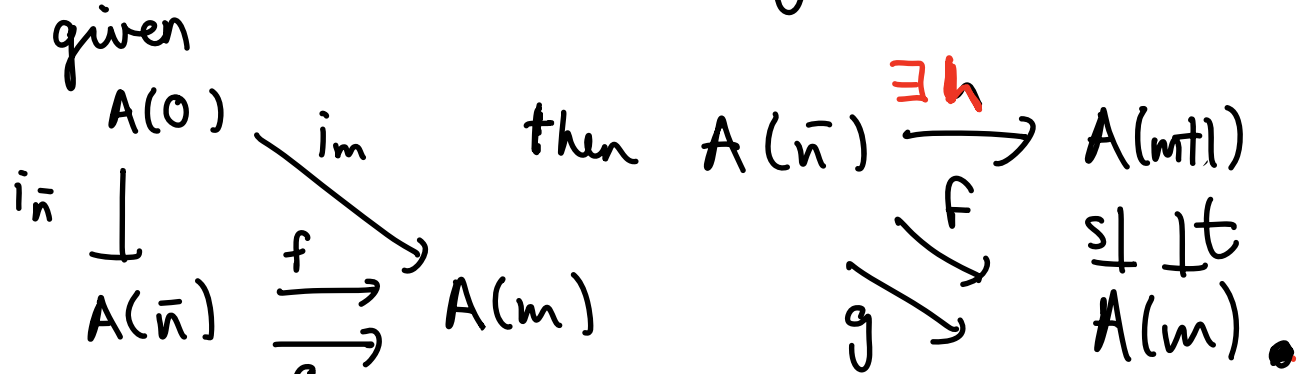
$$p_{k+2}^{\bar{n}^+} \circ i' \circ i_{\bar{n}} \stackrel{\text{def}}{=} i \circ t \circ p_{\bar{n}} \circ i_{\bar{n}}$$

$$\stackrel{\text{def of } i_{\bar{n}} \text{ as limit}}{=} i \circ t \circ i_{n_k} \stackrel{\text{cone}}{=} i \circ i_{n_{k+1}} \stackrel{\text{def}}{=} i_{n_{k+2}}$$

Hence they agree on postcomp. w' the pullback
 projections

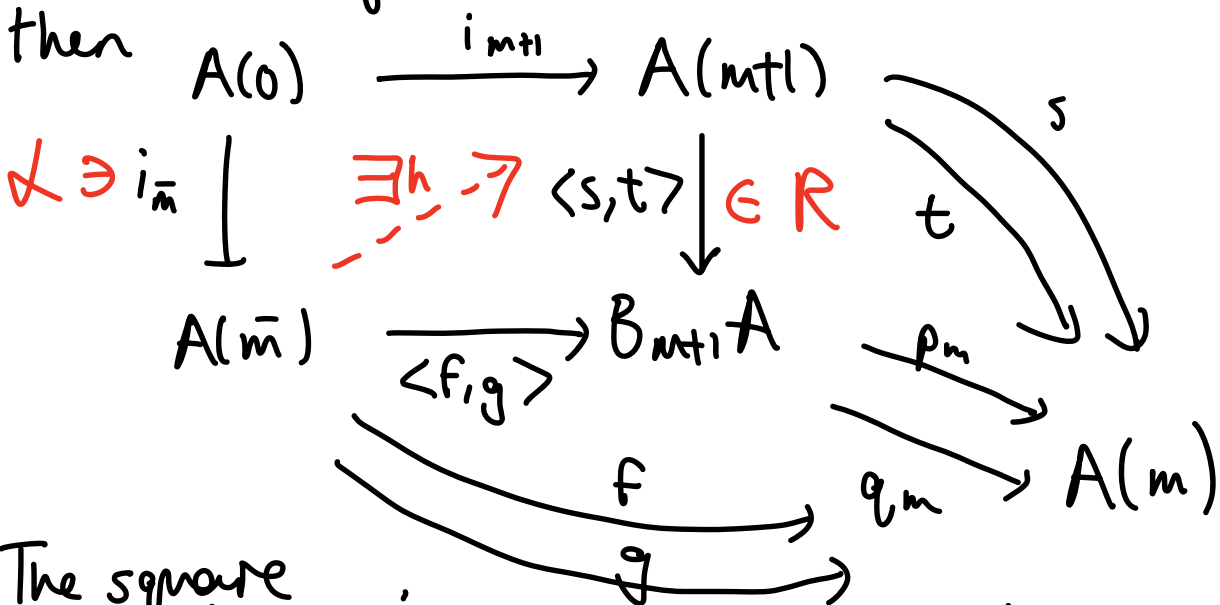
- It remains to prove contractibility of $\text{End}(i/A)$.

- This amounts to showing that



st. $s \circ f = s \circ g$ &
 $t \circ f = t \circ g$

- Get $\langle f, g \rangle : A(\bar{n}) \longrightarrow B_{m+1}A$ &



The square commutes since p_m, q_m are jointly mono. Hence we obtain a diagonal filler

completing the proof . \square

References

- This was proven by Van den Berg & Garner
"Types are weak ω -groupoids"
using Batanin weak ω -cats.
- I wrote a short expository paper
showing how their proof can be done
much more briefly if we use
Grothendieck ω -groupoids,

Note on the construction of globular
weak ω -groupoids from types,
topological spaces etc.

which is what this lecture was
based on .