L8 - Other models of $(\infty,1)$-category

- As mentioned last week there are the classical Quillen model structures on Top & SSet, & a Quillen equivalence

$$\mathrm{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \mathrm{SSet}\;.$$

- Fibrant spaces = all
  fibrant ssets = Kan complexes := simplicial ∞-groupoids.
- So we obtain

$$\mathrm{Ho}(\mathrm{Top}) \simeq \mathrm{Ho}(\mathrm{Kan}) \quad \text{saying}$$

topological spaces $\equiv$ simplicial ∞-groupoids, a form of the homotopy hypothesis, appropriate to simplicial setting.

- What should an $(\infty,1)$-cat be?

  A simple answer:

  a category enriched in ∞-groupoids.

  We can take this to mean topologically or simplicially enriched categories.

  We will take SSet-categories,

which include Kan-enriched cats

- A simplicially enriched cat $\mathcal{C}$ has objects $a, b, c, \ldots$, simplicial sets $\mathcal{C}(a,b)$ — here assumed to be small.
  comp^n $\mathcal{C}(b,c) \times \mathcal{C}(a,b) \to \mathcal{C}(a,c)$,
  ids $1 \longrightarrow \mathcal{C}(a,a)$,
  strictly associative & unital.
- We call the elements of $\mathcal{C}(a,b)_n$ n-morphisms.
- Then for each n, we have a cat $\mathcal{C}_n$ of objects & n-morphisms.
- Moreover the face & degeneracy maps $\cdots\; \mathcal{C}(a,b)_1 \rightleftarrows \mathcal{C}(a,b)_0$ are then id.on.obs functors.

- In this way we can identify simplicially enriched categories with

(2) Functors $\Delta^{op} \longrightarrow Cat_{i.o.}$ where $Cat_{i.o.}$ consists of small categories & identity on objects functors.

There is a third way we will look at later.

Since $\mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{SSet} \xleftarrow{N} \mathrm{Cat}$ preserve products, they induce functors

$$\mathrm{Top}\text{-}\mathrm{Cat} \xrightarrow{\mathrm{Sing}_*} \mathrm{SSet}\text{-}\mathrm{Cat} \xleftarrow{N_*} \mathrm{Cat}\text{-}\mathrm{Cat}$$

- $\mathrm{Sing}_* C$ has same obs as C,
  homs $\mathrm{Sing}_* C(A,B) = \mathrm{Sing}(C(A,B))$
  & comp^n

$$\mathrm{Sing}(C(B,C)) \times \mathrm{Sing}(C(A,B)) \cong \mathrm{Sing}(C(B,C) \times C(A,B))$$
$$\downarrow \mathrm{Sing}(\circ)$$
$$\mathrm{Sing}(C(A,C))$$

- $\mathrm{Sing}_* C$ always enriched in Kan-complexes, i.e. <u>enriched in ∞-groupoids</u>.

$N_* : 2\text{-}Cat = Cat\text{-}Cat \hookrightarrow SSet\text{-}Cat$

identifies 2-Cat as a full subcategory of SSet-Cat containing those SSet-enriched cats $\mathcal{C}$ which are <u>locally nerves of cats</u> -

these are hence locally $(\infty,1)$-cats

(<u>certain $(\infty,2)$-cats</u> - a topic for another day.)

Simplicially-enriched cats vs quasicats

- We would like an adjunction

$$S\text{-}Cat := SSet\text{-}Cat \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} SSet$$

which means we should give

$$\Delta \longrightarrow S\text{-}Cat.$$

- Obvious answer:

$$R : \Delta \hookrightarrow Cat \xrightarrow{D} 2\text{-}Cat \overset{N_*}{\subseteq} S\text{-}Cat$$

where D views a cat as loc. discrete 2-cat.

- But then $S\text{-}Cat(R[n], C) \cong$
$Cat([n], \underset{\text{underlying cat of } C}{UC})$

- Then N_R is just the composite

$$S\text{-}Cat \xrightarrow{U} Cat \xrightarrow{N} SSet$$

$$C \longmapsto UC \text{ where}$$
$$(UC)(a,b) = \underline{C(a,b)_0}.$$

Forgets <u>far too much!</u>

- Need a $\Delta \longrightarrow S\text{-}Cat$ which will encode more info.

- Consider the adjunction

$$\mathrm{Cat} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \text{R-Graph}$$

cat of reflexive graphs

- FX has morphisms -
  - sequences $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$ where each f_i is non-degenerate,
  - $x \xrightarrow{1_x} x$ the chosen degeneracies.

Composition in FX is by
- concatenation / deletion of identities.

- Unit $\eta_X : X \to UFX$ & counit $\varepsilon_c : FUc \to c$ identity-on-objects.
- Comonad $FU \circlearrowright \mathrm{Cat}$ induces @ $c \in \mathrm{Cat}$

$\Delta^{op} \xrightarrow{\mathrm{Res}\, c} \mathrm{Cat}$ its simplicial resolution

$$\cdots\cdots FUFUFUc \; \substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; FUFUc \; \substack{\xrightarrow{\varepsilon_{FUc}} \\ \xleftarrow{F\eta_{Uc}} \\ \xrightarrow[FU\varepsilon_c]{}} \; FUc \quad \&$$

all of these maps are id on obs - so this defines a simplicially-enriched category ResC.

- So n-arrows of ResC are paths of paths of paths ... in C.

- Obtain $\Delta \hookrightarrow \mathrm{Cat} \xrightarrow{\mathrm{Res}} S\text{-}\mathrm{Cat}$ & this is our functor.

- $\mathrm{Res}([0]) = \{ \bullet \}$.
- $\mathrm{Res}([1]) = \{ 0 \xrightarrow{01} 1 \}$ only one non-degen map.
- $\mathrm{Res}([2])$:

$$\begin{tikzcd} & 1 \arrow[dr, "12"] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur, "01"] \arrow[rr, "02"'] & {} & 2 \end{tikzcd}$$

1-arrow

$$[01,12] \xrightarrow{[[01,12]]} [02]$$

This induces our adjunction

$$\begin{tikzcd} S\text{-}\mathrm{Cat} \arrow[r, phantom, "\perp"] & S\mathrm{Set} \arrow[l, bend right, "\mathfrak{C}"'] \arrow[from=1-1, to=1-2, bend right, "H = N_{\mathrm{Res}}"'] \end{tikzcd}$$

homotopy coherent nerve.

Then $HC(2)$ = diagrams $\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\alpha \Downarrow"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c \end{tikzcd}$ in C

where $\alpha : g \cdot f \Rightarrow h$.

If C is a 2-category, viewed as a simplicially-enriched cat, in fact

$HC = N\mathrm{Hom}([n], C)$

set of normal lax functors from $[n] \to C$.

There are model structures on S-Cat & SSet called the Bergner & Joyal model structures respectively whose fibrant objects are the

- Kan enriched cats
- quasicats

& then

$$\begin{tikzcd} S\text{-}Cat \arrow[r, bend right, "H = N_{Res}"'] \arrow[r, phantom, "\perp"] & SSet \arrow[l, bend right, "\mathfrak{C}"'] \end{tikzcd}$$

is a Quillen equivalence, giving a sense in which these provide the same model of homotopy theory.

Segal categories

- Recall our perspective on simplicial enriched cats as functors

  $* \quad \Delta^{op} \xrightarrow{X} Cat$ whose components are all i.o.

- These correspond to internal cats in $SSet = (\Delta^{op}, Set)$

$$X1 \times_{X0} X1 \longrightarrow X1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{i} \\ \xrightarrow[t]{} \end{array} X0$$ with $X0$ discrete.

- If $\mathcal{C}$ is simplicially enriched, the corresp. internal cat in SSet looks like

$$\sum_{a,b,c \in ob C} \mathcal{C}(a,b) \times \mathcal{C}(b,c) \xrightarrow{\circ} \sum_{a,b \in ob C} \mathcal{C}(a,b) \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{i} \\ \xrightarrow[t]{} \end{array} ob C$$

- Evaluating at n, it gives an ordinary cat $\mathcal{C}_n$ - the cat of objects of $\mathcal{C}$ & n-morphisms.

- Now an internal cat in SSet extends !'ly to a functor $\Delta^{op} \xrightarrow{X} SSet$ satisfying the Segal condition.

(Functors $\Delta^{op} \xrightarrow{X} SSet$ will be simplicial spaces)

- So a simplicially-enriched cat C is a simplicial space X such that
① X_0 is discrete.
② X satisfies the Segal condition.

If X satisfies ① it is called a Segal precategory.

- Given a simplicial space X, we always have

$$\boxed{0<1} \xrightarrow{\delta_1^2} \boxed{0<1<2}$$

$$\begin{tikzcd}
 & X2 \arrow[dl, "s_2 = \mathrm{Segal}_2"'] \arrow[dr, "X\delta_1^2"] & \\
X1 \times_{X0} X1 & & X1
\end{tikzcd} \in \mathrm{SSet}$$

- If the Segal condition holds, then composition is encoded by

$$\begin{tikzcd}
 & X2 \arrow[dr, "X\delta_1^2"] & \\
X1 \times_{X0} X1 \arrow[ur, "(s_2)^{-1}"] & & X1
\end{tikzcd}$$

In a Segal cat, we weaken composition.

Def^n) A Segal cat is a Segal precategory such that the Segal maps

$$X_n \xrightarrow{\quad s_n \quad} X1 \times_{X0} X1 \times \dots \times_{X0} X1$$

are weak equivalences $\in$ SSet.

- This kind of weakening can be considered in other contexts as well,

  eg in Cat:

  then a simplicial category

  $X : \Delta^{op} \longrightarrow \mathrm{Cat}$ sat.

  the Segal cond. is a double cat.

- Asking that the Segal maps

  $X_n \longrightarrow X_1 \times_{X_0} \cdots X_1$

  are equivalences of cats corresponds to pseudo-double cats.

- Asking that $X(0)$ is discrete forces all squares in the double cat.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\alpha \Downarrow"] & b \arrow[d, equal] \\ a \arrow[r, "g"'] & b \end{tikzcd}$$

  to have trivial vert. components so it looks globular

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha \Downarrow"] & b \end{tikzcd}$$

& we get bicategories / 2-categories.

- There is a model structure on

$$\mathrm{PreCat} \hookrightarrow [\Delta^{op}, sSet],$$ the

category of Segal precats, whose fibrant objects are the Reedy-Fibrant Segal cats.

This is Quillen equivalent to those earlier described.

I would like to at least say what Reedy-fibrancy means.

- Firstly lets revisit the Segal condition.

- Consider the spine inclusions

$$Sp\Delta^n \hookrightarrow \Delta^n \in (\Delta^{op}, Set)$$

Taking weighted limits of $X : \Delta^{op} \to (\Delta^{op}, Set)$

obtain

$$\begin{array}{ccc} \{\Delta^n, X\} & \longrightarrow & \{Sp\Delta^n, X\} \\ \wr\mid & & \wr\mid \\ X_n & \xrightarrow{\text{segal}} & X_1 \times_{X_0} X_1 \times_{X_0} \cdots X_1 \end{array}$$

so upper horizontal is Segal map.

② Reedy fibrancy

- For each boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ the induced $\{\Delta^n, X\} \longrightarrow \{\partial\Delta^n, X\}$ is a Kan-fibration.

- Since Mono $=$ Cell$\{\partial\Delta^n \hookrightarrow \Delta^n\}$, Reedy-fib. implies $\{V, X\} \to \{U, X\}$ a Kan-fib $\forall\ U \hookrightarrow V$ mono.

- In partic, the Segal maps are then fibrations & so <u>trivial fibrations</u>

Then have

$$
\begin{tikzcd}
 & X2 \arrow[dl, "\text{Segal}"'] \arrow[dr, "X\delta^2_1"] & \\
X1 \times_{X0} X1 & & X1
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[r, "g"] & c
\end{tikzcd}
\quad \longleftarrow \quad
\begin{tikzcd}
 & b \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\
a \arrow[ur, "f"] \arrow[rr, "\exists h"'] & {} & c
\end{tikzcd}
\quad \longrightarrow \quad
\begin{tikzcd}
a \arrow[r, "h"'] & c
\end{tikzcd}
$$

- Also considering, $\emptyset \longrightarrow \Delta^n$, it follows that each X_n is a <u>Kan complex.</u>

Complete Segal spaces

- Here one starts again with simplicial spaces.

$$X : \Delta^{op} \longrightarrow (\Delta^{op}, S).$$

1. We keep the condition that the Segal maps are weak equivalences.
2. We require Reedy fibrancy.

   We drop requirement that X_0 is discrete but add

3. Completeness

   - $J \in Cat$ the free iso $0 \rightleftarrows 1$ & consider $N(J)$, the nerve of the free iso
   - Can think of it as "free equiv in an ∞-cat".

     & $\Delta^0 \hookrightarrow N(J)$ either inclusion.

     (This is a gen. triv cof in Joyal model structure.)

- Completeness means that
$$\{N(J), X\} \longrightarrow \{\Delta^0, X\} = X0$$
is a weak equivalence (equally a t.fib).

- Usually it is formulated as saying that its section (corr to $N(J) \longrightarrow \Delta^0$)
$$X0 \longrightarrow \{N(J), X\}$$
is a weak equivalence.

- The idea is that
$$\{N(J), X\} = hoeq(X) \hookrightarrow X(1)$$
is the object of homotopy equivalences in X.

- This says that the identities map
$$X(0) \longrightarrow hoeq(X) \text{ is an equiv.}$$
It is closely connected to univalence (Stenzel)

Fun Fact (Stenzel)

$X : \Delta^{op} \longrightarrow (\Delta^{op}, Set)$ is a complete Segal sp
$$\Longleftrightarrow$$
$$(\Delta^{op}, Set)^{op} \xrightarrow{\{-, X\}} (\Delta^{op}, Set)$$
is right Quillen functor from Joyal model structure to classical Kan-model str.

There is a model str. on $(\Delta^{op}, SSet)$ whose fibrant obs are the complete Segal spaces, & it is Quillen equivalent to the others.

Summary

4 simplicial models of $(\infty,1)$-cat this week & last.

1. Quasicats
2. Simplicially enriched cats
3. Segal cats
4. Complete Segal spaces.

All equivalent, via equivalences of model cats.