

Lecture 9 - A model-independent approach to $(\infty, 1)$ -categories

• Last time :

different models of $(\infty, 1)$ -category :

- ① quasicats
- ② Segal cats
- ③ complete Segal spaces
- ④ simplicially enriched cats

• Would be nice to do " ∞ -category theory" (i.e. adjoints, limits etc) in a way that is independent of which definition we use.

• This is the idea of Riehl & Verity's ∞ -cosmoi.

• Only applies to 1, 2 & 3 above. Simpl. enriched cats have some problems because of their semi-strict nature (e.g. the maps between them are strict) which make it problematic working with them.

• We will approach the notion of an ∞ -cosmos gradually.

The 2-category of quascategories

- Let $\mathcal{Q}\text{Cat} \hookrightarrow \mathcal{S}\text{Set}$ denote the full subcat of quascategories.

- It is also cartesian closed:

① since q-cats are the fibrant obs, closed under products & 1

② - If B is a quasicat, then $\mathcal{S}\text{Set}(A, B)$ a quasicat. In partic, $\mathcal{Q}\text{Cat}(A, B)$ a quasicat.

- Have adjunction $\text{Cat} \begin{array}{c} \xleftarrow{h} \\ \perp \\ \xrightarrow{N} \end{array} \mathcal{Q}\text{Cat}$

as described in L7:

hX has same objects as X & arrows: homotopy-classes $a \xrightarrow{[F]} b$.

- In fact h preserves finite products (follows from this description or since $\mathcal{Q}\text{Cat}$ "exponential ideal" in $\mathcal{S}\text{Set}$).

- So get 2-Cat $\begin{array}{c} \xleftarrow{H = h_*} \\ \perp \\ \xrightarrow{N_*} \end{array} \mathcal{Q}\text{Cat-Cat}$

$$\text{Z-cat} \begin{array}{c} \xleftarrow{H = h_*} \\ \perp \\ \xrightarrow{N_*} \end{array} \text{QCat-cat}$$

- For \mathcal{C} enriched in quasicoats, $H\mathcal{C}$ a Z-cat:
 - objects as in \mathcal{C} ,
 - arrows: the objects of $\mathcal{C}(a, b)$
 - Z-cells: homotopy classes of arrows in $\mathcal{C}(a, b)$.
- In other words, $H\mathcal{C}$ has same underlying cat as \mathcal{C} and homotopy classes of Z-cells.

Now QCat is QCat -enriched, so can form $h\text{QCat}$ - the Z-category of quasicoats:

- obs, arrows as in QCat (∞ -cats, ∞ -functors)
- Z-cells homotopy classes of " ∞ -nat t's".

A 2-categorical approach to quasicons

Defⁿ) An adjunction / equivalence of quasicons is an adjunction / equiv. in the 2-category hQCat.

- In el. terms, this means

$$A \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{U} \end{array} B +$$

$$A \begin{array}{c} \xrightarrow{F} B \\ \varepsilon \Downarrow \\ \xrightarrow{1} A \end{array} \begin{array}{c} \xrightarrow{U} \\ \Downarrow \\ \end{array} A \quad \& \quad B \begin{array}{c} \xrightarrow{1} B \\ \eta \Downarrow \\ \xrightarrow{f} A \end{array} \begin{array}{c} \xrightarrow{U} \\ \Downarrow \\ \end{array} B$$

sat. triangle equations

$$1 \xrightarrow{F\eta} 1 \xrightarrow{UF} F \quad \& \quad U \xrightarrow{\eta U} U \xrightarrow{U\varepsilon} U$$

$\underbrace{\hspace{10em}}_1 \quad \underbrace{\hspace{10em}}_1$

- Note: this really means that the eq's hold up to homotopy in $\mathcal{Q}\text{-Cat}(A, A), \mathcal{Q}\text{-Cat}(B, B)$ - since looking at homotopy-classes of 2-cells.

- Surprising thing: this captures correct notion of adjunction between ∞ -cats.

Corollary: Adjoints & equivalences can be composed (etc).

Proof) Use usual 2-categorical argument in \mathbf{hQCat} - no " ∞ -arguments" needed.

∞ -cosmoi Version 0

- Now it turns out that the categories
 - CSS of complete Segal spaces
 - SegCat of Segal categoriesare naturally simplicially enriched - indeed there are product preserving functors
$$K: \text{CSS}, \text{SegCat} \longrightarrow \text{SSet}$$
taking values in quasicats, so can define $\text{CSS}(A, B) = \text{QCat}(KA, KB)$.
- Since $\text{CSS}, \text{SegCat}$ are QCat -enriched, can form homotopy 2-cats $H(\text{CSS}), H(\text{SegCat})$ & again these capture the correct notions of adjunction & equivalence, analytically defined -
 - ie. the elementary definitions one uses in the specific context, using things like initial obs is "slice ∞ -cats".

∞ -cosmoi Version 0 ctd

- So if all of " ∞ -category theory" we care about are adjunctions & equivalences,

Def VO) An ∞ -cosmos \mathcal{C} is a QCat-enriched category.

Will call objects of \mathcal{C} " ∞ -cats"
& prove things about them using the 2-category \mathbf{HE} .

But ∞ -category theory should also concern structures like limits in ∞ -cats, & for these the defⁿ above is not enough.

- Limits in an ω -cat A have diagram shape J a simplicial set (not nec. ω -cat)
- So can't consider $D: J \longrightarrow A$ as a morphism in $\mathcal{Q}\text{Cat}$.
- But $[J, A] = \text{SSet}(J, A) \in \mathcal{Q}\text{Cat}$ is the power of A by J in $\mathcal{Q}\text{Cat}$ -
ie. $\mathcal{Q}\text{Cat}(B, [J, A]) \cong \text{SSet}(J, \mathcal{Q}\text{Cat}(B, A))$
a certain kind of defining nat iso weighted limit.

Axiom (Powers)

An ω -cosmos \mathcal{C} has powers by simplicial sets.

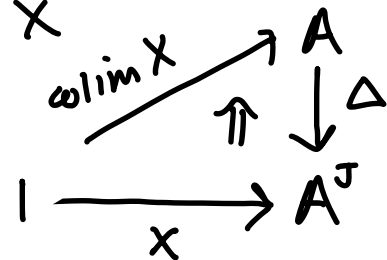
- Then given $A \in \mathcal{C}$, we can form the power $A^J \in \mathcal{C}$
- Taking powers is functorial in SSet -
in partic, the map $J \longrightarrow 1 \in \text{SSet}$ induces,
by the univ. prop. of powers, a diagonal
map $\Delta: A \longrightarrow A^J$.

Defⁿ) A has J -lims if Δ has a right adj
 J -colims if Δ has a left adj.

What if we want to capture the colimit of a particular diagram?

- In $\mathcal{Q}\mathcal{C}\mathcal{A}\mathcal{T}$, can capture a diagram as $I \xrightarrow{X} A^J$ as I is a $\mathcal{Q}\mathcal{C}\mathcal{A}\mathcal{T}$.

- Then can capture limit of X

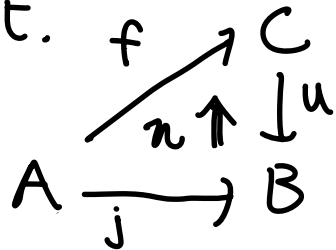


as the right adjoint to Δ relative to X .

- What is a relative adjunction in a 2-category?

Firstly, what is a relative adjunction in $\mathcal{C}\mathcal{A}\mathcal{T}$?

Given a functor $j: A \rightarrow B$ & $u: C \rightarrow B$ we call $F: A \rightarrow C$ left adjoint to u relative to j when there is a nat η .



such that $\mathcal{C}(Fx, y) \xrightarrow{\quad} \mathcal{C}(jx, uy)$
 $Fx \xrightarrow{\alpha} y \xrightarrow{\quad} jx \xrightarrow{\eta x} ufx \xrightarrow{\nu ux} uy$
 a bijection.

(Equivalently, for each $a \in A$, $\exists ja \xrightarrow{\eta a} ua$ u -universal.)

write $F \dashv_j u$ & say F is j -left adjoint to u .

- In a 2-cat \mathcal{C} we say

$$\begin{array}{ccc} & f \nearrow & C \\ & n \uparrow & \downarrow u \\ A & \xrightarrow{j} & B \end{array}$$

exhibits f as j -left adj to u

if $\forall D \in \mathcal{C}$

$$\begin{array}{ccc} & \mathcal{C}(D, C) & \text{exhibits} \\ & f_* \nearrow & \downarrow u_* \\ \mathcal{C}(D, A) & \xrightarrow{j_*} & \mathcal{C}(D, B) \end{array} \quad \begin{array}{l} f_* \dashv j_* \quad u_* \\ \text{in Cat.} \end{array}$$

- l. define concept representably.

- In elementary terms,

given

$$\begin{array}{ccc} D & \xrightarrow{y} & C \\ x \downarrow & \alpha \nearrow & \downarrow u \\ A & \xrightarrow{j} & B \end{array} \quad \exists! \quad \begin{array}{ccc} D & \xrightarrow{y} & C \\ x \downarrow & \beta \nearrow & \downarrow u \\ A & \xrightarrow{j} & B \end{array} \quad \text{st}$$

$$\begin{array}{ccc} D & \xrightarrow{y} & C \\ x \downarrow & \beta \nearrow & \downarrow u \\ A & \xrightarrow{j} & B \end{array} \quad \begin{array}{ccc} D & \xrightarrow{y} & C \\ \beta \uparrow & f \nearrow & \downarrow u \\ A & \xrightarrow{j} & B \end{array} = \alpha.$$

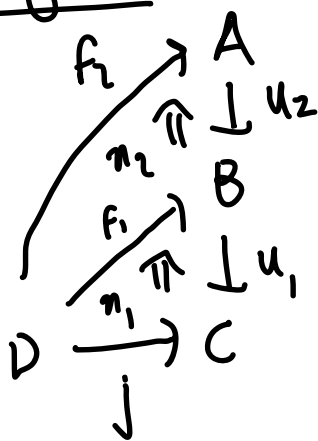
Remarks). Also j -left adjoints are called absolute left liftings.

left lifting $\begin{array}{ccc} & f \nearrow & C \\ & n \uparrow & \downarrow u \\ A & \xrightarrow{j} & B \end{array}$ & $D \xrightarrow{g} A \begin{array}{ccc} & f \nearrow & C \\ & n \uparrow & \downarrow u \\ A & \xrightarrow{j} & B \end{array}$ left lifting.

- Now we define a relative adjunction $f \dashv_j u$ in a QCat-enriched \mathcal{C} to be a relative adjunction in $h\mathcal{C}$.

- Right adjoints relative to j defined dually - reverse 2-cells.

Pasting Lemma



Suppose $f_1 \dashv_j u_1$.
Then $f_2 \dashv_{f_1} u_2 \Leftrightarrow f_2 \dashv_j u_2 u_1$.

Proof] Sim. to standard lemma for Kan extensions.

Colimits in an ω -cat

- One more thing: in an ω -cosmos \mathcal{C} we don't consider diagrams as morphisms $I \rightarrow A^J$.

Eg. in the ω -cosmos of pted ω -cats, \exists only one such diagram.

Hence must allow "diagrams" $B \rightarrow A^J$ for B arbitrary.

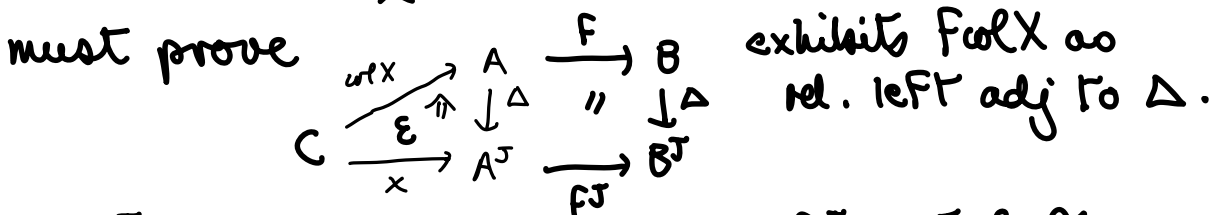
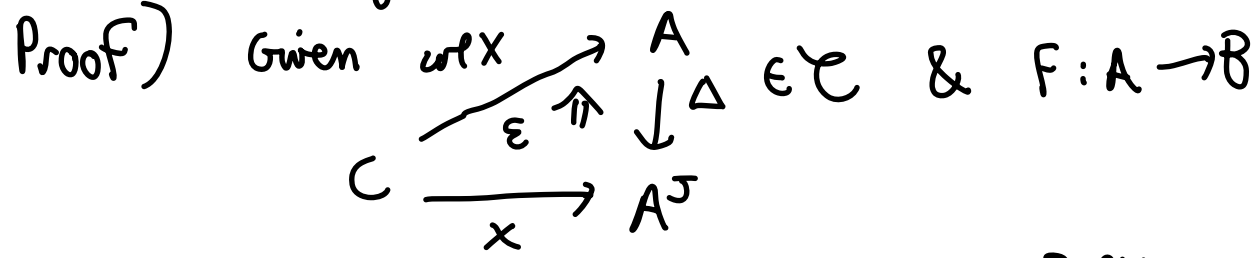
Definition) let \mathcal{C} be an ω -cosmos.

A colimit of $B \xrightarrow{X} A^J$ is a left adjoint to X relative to Δ :

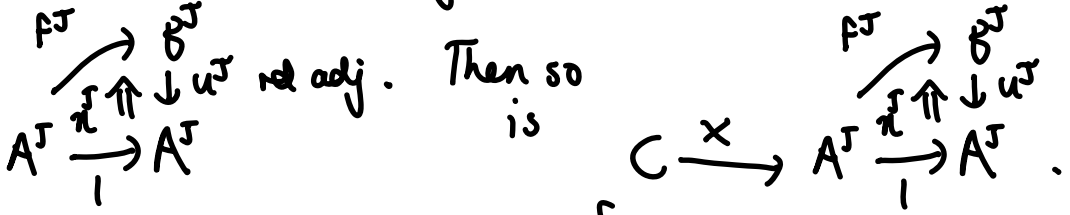
$$\begin{array}{ccc} \text{col } X & \rightarrow & A \\ & \Uparrow & \downarrow \Delta \\ B & \xrightarrow{X} & A^J \end{array} .$$

Remark) In $\mathcal{Q}\text{Cat}$, CSS , SegCat suffices to look at diagrams with $B = I$.

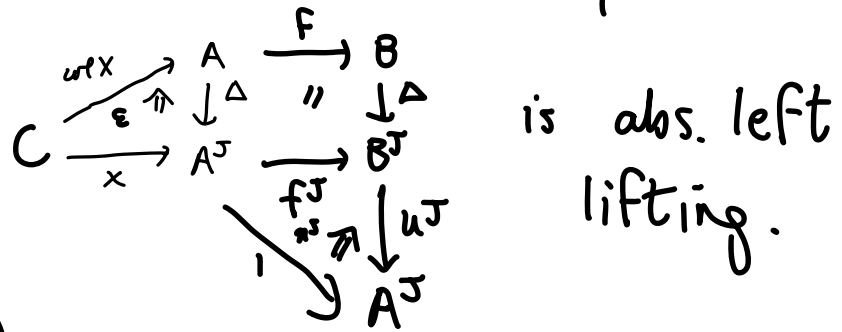
Thm) Left adjoints preserve colims.



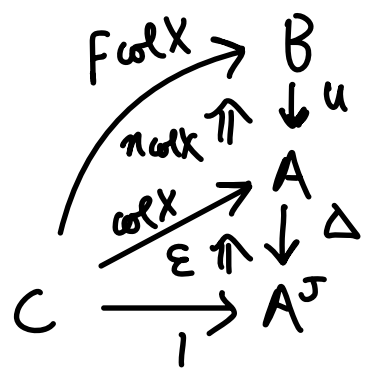
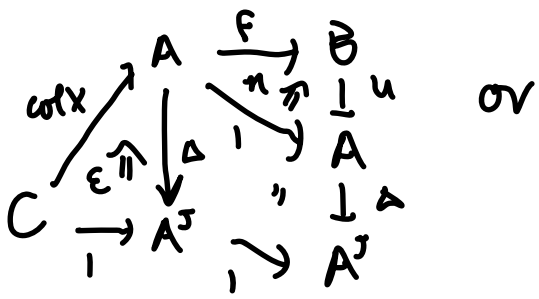
Now $(-)^J$ preserves adjunctions, so $F^J \dashv u^J$ & then



So by Lemma, suff to show



But this equals



Further aspects of ∞ -cat theory

- For a morphism $F: A \rightarrow B$ of ∞ -cats, certainly we would like to form comma ∞ -cat B/F For instance.
- If $B^{\Delta(1)}$ denotes the ∞ -cat of arrows, we have $B^{\Delta(1)} \xrightarrow{\text{cod}} B$ induced by restriction along $\Delta(0) \xrightarrow{\delta_0'} \Delta(1)$.
- Now can form

pullback

$$\begin{array}{ccc}
 B/F & \longrightarrow & B^{\Delta(1)} \\
 \downarrow & \lrcorner & \downarrow \text{cod} \\
 A & \xrightarrow{F} & B
 \end{array}
 \quad
 \begin{array}{ccc}
 (a, b \xrightarrow{K} fa) & \mapsto & (b \rightarrow fa) \\
 \downarrow & & \downarrow \\
 a & \mapsto & fa
 \end{array}$$

Problem) $\mathcal{Q}\text{Cat} \hookrightarrow \text{SSet}$ not closed under pbs.

However $\text{cod}: B^{\Delta(1)} \longrightarrow B$ is a fibration

& $\mathcal{Q}\text{Cat}$ closed under pbs of fibrations.

- In fact $\mathcal{Q}\text{Cat}$, CSS & SegCat all arise as fibrant objects in model cats, so come with natural class of maps: the fibrations between fibrant objects.
- In $\mathcal{Q}\text{Cat}$, these are the maps with lifting prop against inner horns & $1 \longrightarrow N(J)$, & are called isofibrations.

Complete definition of ω -cosmos

A $\mathcal{Q}at$ enriched cat \mathcal{C} equipped with a class of maps $A \rightarrow B$ called isofibrations.

These satisfy the following axioms:

Limits

- ① \mathcal{C} has powers, all small products, pullbacks of isofibrations & limits of countable towers of isofibrations.

Behavior of isofibrations

- ② $A \rightarrow 1$ an isofib.
- ③ If $A \rightarrow B$ an isofib, then $\mathcal{C}(C, A) \rightarrow \mathcal{C}(C, B)$ isofib. of $qats$.
- ④ Isofibrations closed under above lim. constructions.
- ⑤ - If $X \rightarrow Y$ mono $\in \mathcal{S}et$, then $A^X \rightarrow A^Y$ isofib. Moreover, if $A \xrightarrow{f} B$ isofib, then

$$\begin{array}{ccc} A^Y & \xrightarrow{f^Y} & B^Y \\ \downarrow A_j & \searrow & \downarrow B_j \\ A^X & \xrightarrow{f^X} & B^X \end{array} \quad \text{an isofib}$$

How do these axioms help us?

Certainly $\Delta^0 \xrightarrow{\delta_0} \Delta^1$ is mono

$\Rightarrow A^{\Delta^1} \xrightarrow{\text{cod}} A^{\Delta^0} = A$ is isofib.

Hence
pullback
exists

$$\begin{array}{ccc} B/\downarrow & \longrightarrow & B^{\Delta(1)} \\ \downarrow & \lrcorner & \downarrow \text{cod} \\ A & \xrightarrow{F} & B \end{array}$$

so we can talk about comma κ -cats,
slices etc, & lots of other
things.

Lots more stuff to be figured
out

Eg. - Coherent, monoidal κ -cats?