Lecture 9 –
A model-independent approach to $(\infty,1)$-categories

- Last time:
  different models of $(\infty,1)$-category:
  1. quasicats
  2. Segal cats
  3. complete Segal spaces
  4. simplicially enriched cats

- Would be nice to do "∞-category theory" (ie. adjoints, limits etc") in a way that is independent of which definition we use.

- This is the idea of Riehl & Verity's <u>∞-cosmoi</u>.

- Only applies to 1, 2 & 3 above. Simpl. enriched cats have some problems because of their semi-strict nature (eg. the maps between them are strict) which make it problematic working with them.

- We will approach the notion of an ∞-cosmos <u>gradually</u>.

The 2-category of quasicategories

- Let $QCat \hookrightarrow SSet$ denote the full subcat of quasicategories.
- It is also cartesian closed:
  1. since q-cats are the fibrant obs, closed under products & 1
  2. - If B is a quasicat, then $SSet(A,B)$ a quasicat. In partic, $QCat(A,B)$ a quasicat.
- Have adjunction
$$\begin{tikzcd} Cat \arrow[r, bend right, "N"'] \arrow[r, phantom, "\perp"] & QCat \arrow[l, bend right, "h"'] \end{tikzcd}$$
as described in L7:

  hX has same objects as X & arrows: homotopy-classes $a \xrightarrow{[f]} b$.
- In fact h preserves finite products (follows from this description or since QCat "exponential ideal" in SSet).

- So get
$$\begin{tikzcd} 2\text{-}Cat \arrow[r, bend right, "N_*"'] \arrow[r, phantom, "\perp"] & QCat\text{-}Cat \arrow[l, bend right, "H = h_*"'] \end{tikzcd}$$

$$\begin{tikzcd} 2\text{-}\mathrm{Cat} \arrow[r, shift right, "N_*"'] \arrow[r, phantom, "\perp"] & \mathrm{QCat}\text{-}\mathrm{Cat} \arrow[l, shift right, "H = h_*"'] \end{tikzcd}$$

- For $\mathcal{C}$ enriched in quasicats, $H\mathcal{C}$ a 2-cat:
  - objects as in $\mathcal{C}$,
  - arrows: the objects of $\mathcal{C}(a,b)$
  - 2-cells: homotopy classes of arrows in $\mathcal{C}(a,b)$.
- In other words, $H\mathcal{C}$ has same underlying cat as $\mathcal{C}$ and homotopy classes of 2-cells.

Now QCat is QCat-enriched, so can form hQCat - the 2-category of quasicats:

- obs, arrows as in QCat (∞-cats, ∞-functors)
- 2-cells homotopy classes of "∞-nat t's".

A 2-categorical approach to quasicats

Def^n) An adjunction / equivalence of quasicats is an adjunction / equiv. in the 2-category $h\mathcal{Q}Cat$.

- In el. terms, this means

$$\begin{tikzcd} A \arrow[r, shift right, "u"'] & B \arrow[l, shift right, "f"'] \end{tikzcd} \quad +$$

$$\begin{tikzcd} & B \arrow[dr, "u"] & \\ A \arrow[ur, "f"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=3, "\varepsilon \Downarrow"] & & A \end{tikzcd} \quad \& \quad \begin{tikzcd} B \arrow[dr, "f"'] \arrow[rr, bend left, "1"] \arrow[rr, phantom, "\eta \Downarrow"] & & B \\ & A \arrow[ur, "u"'] & \end{tikzcd}$$

sat. triangle equations

$$\begin{tikzcd} f \arrow[r, "f\eta"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, shift right=3, "="] & fuf \arrow[r, "\varepsilon f"] & f \end{tikzcd} \quad \& \quad \begin{tikzcd} u \arrow[r, "\eta u"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, shift right=3, "="] & ufu \arrow[r, "u\varepsilon"] & u \end{tikzcd} .$$

- Note: this really means that the eq.'s hold up to homotopy in $\mathcal{Q}\text{-}Cat(A,A)$, $\mathcal{Q}\text{-}Cat(B,B)$ - since looking at homotopy-classes of 2-cells.

- Surprising thing: this captures correct notion of adjunction between ∞-cats.

Corollary: Adjoints & equivalences can be composed (etc).

Proof) Use usual 2-categorical argument in $h\mathcal{Q}\mathrm{Cat}$ - no "∞-arguments" needed.

∞-cosmoi Version 0

- Now it turns out that the categories
  - CSS of complete Segal spaces
  - SeCat of Segal categories

  are naturally simplicially enriched - indeed there are product preserving functors

  $K : \mathrm{CSS}, \mathrm{SeCat} \longrightarrow \mathrm{SSet}$

  taking values in quasicats, so can define $\mathrm{CSS}(A,B) = \mathrm{QCat}(KA, KB)$.
- Since CSS, SeCat are QCat-enriched, can form homotopy 2-cats $H(\mathrm{CSS}), H(\mathrm{SeCat})$ & again these capture the correct notions of adjunction & equivalence, analytically defined -

ie. the elementary definitions one uses in the specific context, using things like initial obs in "slice ∞-cats".

∞-cosmoi Version 0 ctd

- So if all of "∞-category theory" we care about are adjunctions & equivalences,

Def V0) An ∞-cosmos $\mathcal{C}$ is a QCat-enriched category.

Will call objects of $\mathcal{C}$ "∞-cats" & prove things about them using the 2-category $H\mathcal{C}$.

But ∞-category theory should also concern structures like limits in ∞-cats, & for these the def^n above is not enough.

- Limits in an ∞-cat A have diagram shape J a simplicial set (not nec. ∞-cat)
- So can't consider $D : J \longrightarrow A$ as a morphism in $\mathcal{Q}Cat$.
- But $[J, A] = SSet(J, A) \in \mathcal{Q}Cat$ is the power of A by J in $\mathcal{Q}Cat$ –
ie. $\mathcal{Q}Cat(B, [J, \mathbf{A}]) \cong SSet(J, \mathcal{Q}Cat(B, A))$

defining nat iso

a certain kind of weighted limit.

Axiom (Powers)

An ∞-cosmos $\mathcal{C}$ has powers by simplicial sets.

- Then given $A \in \mathcal{C}$, we can form the power $A^J \in \mathcal{C}$
- Taking powers is functorial in $SSet$ – in partic, the map $J \longrightarrow 1 \in SSet$ induces, by the univ. prop. of powers, a diagonal map $\Delta : A \longrightarrow A^J$.

Def^n) A has J-lims if Δ has a right adj
J-colims if Δ has a left adj.

What if we want to capture the colimit of a particular diagram?

- In QCat, can capture a diagram as $1 \xrightarrow{X} A^J$ as 1 is a qcat.
- Then can capture limit of X

$$
\begin{tikzcd}
 & A \arrow[d, "\Delta"] \\
1 \arrow[ur, "\mathrm{colim}\, X"] \arrow[r, "X"'] & A^J \arrow[u, phantom, shift right=4, "\Uparrow"]
\end{tikzcd}
$$

as the right adjoint to Δ relative to X.

- What is a relative adjunction in a 2-category?

Firstly, what is a relative adjunction in Cat?

Given a functor $j : A \longrightarrow B$ & $u : C \longrightarrow B$ we call $f : A \longrightarrow C$ left adjoint to u relative to j when there is a nat t.

$$
\begin{tikzcd}
 & C \arrow[d, "u"] \\
A \arrow[ur, "f"] \arrow[r, "j"'] & B \arrow[u, phantom, shift right=4, "\eta \Uparrow"]
\end{tikzcd}
$$

such that $C(fx, y) \longrightarrow B(jx, uy)$

$fx \xrightarrow{\alpha} y \longmapsto jx \xrightarrow{\eta_x} ufx \xrightarrow{u\alpha} uy$

a bijection.

(Equivalently, for each $a \in A$, $\exists\ ja \xrightarrow{\eta_a} ua'$ u-universal.)

Write $f \dashv_j u$ & say f is j-left adjoint to u.

- In a 2-cat $\mathcal{C}$ we say

$$\begin{tikzcd} & C \arrow[d, "u"] \\ A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, "\eta\Uparrow", shift right=3] & B \end{tikzcd}$$

exhibits f as j-left adj to u

if $\forall D \in \mathcal{C}$

$$\begin{tikzcd} & \mathcal{C}(D,C) \arrow[d, "u_*"] \\ \mathcal{C}(D,A) \arrow[ur, "f_*"] \arrow[r, "j_*"'] \arrow[ur, phantom, "\eta_*\Uparrow", shift right=3] & \mathcal{C}(D,B) \end{tikzcd}$$

exhibits $f_* \dashv_{j_*} u_*$ in Cat.

- ie. define concept representably.
- In elementary terms,

given

$$\begin{tikzcd} D \arrow[r, "y"] \arrow[d, "x"'] \arrow[dr, phantom, "\alpha \Nearrow"] & C \arrow[d, "u"] \\ A \arrow[r, "j"'] & B \end{tikzcd}$$

$\exists !$

$$\begin{tikzcd} D \arrow[r, "y"] \arrow[d, "x"'] \arrow[dr, phantom, "\beta\Uparrow", shift left=2] & C \\ A \arrow[ur, "f"'] & \end{tikzcd}$$

st

$$\begin{tikzcd} D \arrow[r, "y"] \arrow[d, "x"'] \arrow[dr, phantom, "\beta\Uparrow", shift left=3] & C \arrow[d, "u"] \\ A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, "\eta\Uparrow", shift right=3] & B \end{tikzcd} = \alpha.$$

Remarks) • Also j-left adjoints are called absolute left liftings –

left lifting

$$\begin{tikzcd} & C \arrow[d, "u"] \\ A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, "\eta\Uparrow", shift right=3] & B \end{tikzcd}$$

&

$$\begin{tikzcd} & & C \arrow[d, "u"] \\ D \arrow[r, "g"] & A \arrow[ur, "f"] \arrow[r, "j"'] \arrow[ur, phantom, "\eta\Uparrow", shift right=3] & B \end{tikzcd}$$

left lifting.

- Now we define a relative adjunction $f \dashv_j u$ in a $\mathcal{Q}\mathcal{C}at$-enriched $\mathcal{C}$ to be a relative adjunction in $h\mathcal{C}$.

- Right adjoints relative to j defined dually - reverse 2-cells.

Pasting Lemma

$$
\begin{tikzcd}
& A \arrow[d, "u_2", "\perp"'] \\
& B \arrow[d, "u_1", "\perp"'] \\
D \arrow[r, "j"'] \arrow[ur, "f_1"] \arrow[uur, bend left, "f_2"] \arrow[ur, phantom, shift right=3, "\Uparrow \eta_1"'] \arrow[uur, phantom, shift right=3, "\Uparrow \eta_2"'] & C
\end{tikzcd}
$$

Suppose $f_1 \dashv_j u_1$.

Then $f_2 \dashv_{f_1} u_2 \iff f_2 \dashv_j u_2 u_1$.

Proof Sim. to standard lemma for Kan extensions.

Colimits in an ∞-cat

- One more thing: in an ∞-cosmos $\mathcal{C}$ we don't consider diagrams as morphisms $1 \longrightarrow A^J$.

Eg. in the ∞-cosmos of pted ∞-cats, $\exists$ only one such diagram.
Hence must allow "diagrams" $B \longrightarrow A^J$ for B arbitrary.

Definition) Let $\mathcal{C}$ be an ∞-cosmos.
A colimit of $B \xrightarrow{X} A^J$ is a left adjoint to X relative to Δ:

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ B \arrow[ur, "\mathrm{col}X"] \arrow[r, "X"'] \arrow[r, phantom, shift left=3, "\Uparrow"] & A^J \end{tikzcd}.$$

Remark) In QCat, CSS, SegCat suffices to look at diagrams with $B = 1$.

Thm) Left adjoints preserve colims.

Proof) Given
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ C \arrow[ur, "\mathrm{col}X"] \arrow[r, "X"'] \arrow[ur, phantom, bend right, "\varepsilon\Uparrow"] & A^J \end{tikzcd}$$
$\in \mathcal{C}$ & $F : A \to B$

must prove
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "F"] \arrow[dr, phantom, "="] & B \arrow[d, "\Delta"] \\ C \arrow[ur, "\mathrm{col}X"] \arrow[r, "X"'] \arrow[ur, phantom, bend right, "\varepsilon\Uparrow"] & A^J \arrow[r, "F^J"'] & B^J \end{tikzcd}$$
exhibits $F\mathrm{col}X$ as rel. left adj to Δ.

Now $(-)^J$ preserves adjunctions, so $F^J \dashv u^J$ & then
$$\begin{tikzcd} & B^J \arrow[d, "u^J"] \\ A^J \arrow[ur, "F^J"] \arrow[r, "1"'] \arrow[ur, phantom, bend right, "\eta^J\Uparrow"] & A^J \end{tikzcd}$$
rel adj. Then so is
$$\begin{tikzcd} & & B^J \arrow[d, "u^J"] \\ C \arrow[r, "X"] & A^J \arrow[ur, "F^J"] \arrow[r, "1"'] \arrow[ur, phantom, bend right, "\eta^J\Uparrow"] & A^J \end{tikzcd}.$$

So by Lemma, suff to show
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "F"] \arrow[dr, phantom, "="] & B \arrow[d, "\Delta"] \\ C \arrow[ur, "\mathrm{col}X"] \arrow[r, "X"'] \arrow[ur, phantom, bend right, "\varepsilon\Uparrow"] & A^J \arrow[r, "F^J"] \arrow[dr, "1"'] \arrow[dr, phantom, bend left, "\eta^J\Uparrow"] & B^J \arrow[d, "u^J"] \\ & & A^J \end{tikzcd}$$
is abs. left lifting.

But this equals
$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[r, "F"] \arrow[dr, "1"'] \arrow[dr, phantom, bend left, "\eta\Uparrow"] & B \arrow[d, "u"] \\ C \arrow[ur, "\mathrm{col}X"] \arrow[r, "1"'] \arrow[ur, phantom, bend right, "\varepsilon\Uparrow"] & A^J \arrow[dr, "1"'] \arrow[r, phantom, "="] & A \arrow[d, "\Delta"] \\ & & A^J \end{tikzcd}$$
or
$$\begin{tikzcd} & B \arrow[d, "u"] \\ & A \arrow[d, "\Delta"] \\ C \arrow[uur, bend left, "F\mathrm{col}X"] \arrow[ur, "\mathrm{col}X"] \arrow[r, "1"'] \arrow[ur, phantom, bend right, "\varepsilon\Uparrow"] \arrow[u, phantom, "\eta\mathrm{col}X\Uparrow"] & A^J \end{tikzcd}$$

Further aspects of ∞-cat theory

- For a morphism $F : A \to B$ of ∞-cats, certainly we would like to form comma ∞-cat B/F for instance.
- If $B^{\Delta(1)}$ denotes the ∞-cat of arrows, we have $B^{\Delta(1)} \xrightarrow{\text{cod}} B$ induced by restriction along $\Delta(0) \xrightarrow[\delta_0^1]{} \Delta(1)$.
- Now can form

pullback
$$\begin{tikzcd} B/f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B^{\Delta(1)} \arrow[d, "\text{cod}"] \\ A \arrow[r, "F"'] & B \end{tikzcd} \qquad \begin{tikzcd} (a, b \xrightarrow{\alpha} Fa) \arrow[r, mapsto] \arrow[d, mapsto] & (b \to Fa) \arrow[d, mapsto] \\ a \arrow[r, mapsto] & Fa \end{tikzcd}$$

Problem) $QCat \hookrightarrow SSet$ not closed under pbs.

However $\text{cod} : B^{\Delta 1} \longrightarrow B$ is a fibration & QCat closed under pbs of fibrations.

- In fact QCat, CSS & SegCat all arise as fibrant objects in model cats, so come with natural class of maps: the fibrations between fibrant objects.

- In QCat, these are the maps with lifting prop against inner horns & $1 \longrightarrow N(J)$, & are called isofibrations.

Complete definition of ∞-cosmos

A $\mathcal{QC}at$ enriched cat $\mathcal{C}$ equipped with a class of maps $A \twoheadrightarrow B$ called isofibrations.

These satisfy the following axioms:

Limits

① $\mathcal{C}$ has powers, all small products, pullbacks of isofibrations & limits of countable towers of isofibrations.

Behavior of isofibrations

② $A \twoheadrightarrow 1$ an isofib.

③ If $A \twoheadrightarrow B$ an isofib, then $\mathcal{C}(C,A) \twoheadrightarrow \mathcal{C}(C,B)$ isofib. of qcats.

④ Isofibrations closed under above lim. constructions.

⑤ . If $X \xrightarrow{i} Y$ mono $\in$ sSet, then $A^Y \twoheadrightarrow A^X$ isofib

Moreover, if $A \xrightarrow{f} B$ isofib, then

$$\begin{tikzcd}
A^Y \arrow[dr, two heads] \arrow[rr, "f^Y"] \arrow[ddr, bend right, two heads, "A^j"'] & & B^Y \arrow[dd, two heads, "B^j"] \\
& \cdot \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \\
& A^X \arrow[r, "f^X"] & B^X
\end{tikzcd}$$

an isofib

How do these axioms help us?

Certainly $\Delta^0 \xrightarrow{\delta^1_0} \Delta^1$ is mono

$\Rightarrow \quad A^{\Delta^1} \overset{\text{cod}}{\twoheadrightarrow} A^{\Delta^0} = A$ is isofib.

Hence pullback exists

$$\begin{tikzcd} B/f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B^{\Delta(1)} \arrow[d, "\text{cod}"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

so we can talk about comma ∞-cats, slices etc, & lots of other things.

Lots more stuff to be figured out

Eg. - Coherent, monoidal ∞-cats?