Lecture 11

- I will now turn to the "marked" simplicial examples.

- A few such have been developed. The first such model are the complicial sets, developed by Verity, a model for (∞,∞)-cats.

- Recently Lurie & others have been looking at a similar model for $(\infty,2)$-cats – called ∞-bicategories. These are very similar & motivated by the complicial sets – so we will study the complicial sets.

Idea

- Want non-invertible 2-simplicies

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}$$

so that we can capture nerves of 2-cats etc.

- But also need the 2-simplices to encode composition of 1-cells

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\mathrm{S1}"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}$$

& such 2-simplices should be "equivalences".

- Therefore, we need to keep track of a collection of "thin" n-simplices, thought of as equivalences.
- A stratified simplicial set X is a simplicial set with a subset of thin n-simplices containing the degeneracies $\forall n \geq 1$.
  Morphisms of stratified simplicial sets preserve thinness.
- Strat = cat of stratified simp. sets

Def) • $f: X \to Y \in$ Strat is regular if $a \in X_n$ is thin $\Leftrightarrow$ Fa is thin.

• $f: X \to Y$ is entire if it is the identity on underlying simpl. sets.

Write $F: X \to_r Y$ & $F: X \to_e Y$ to indicate F is regular/entire.

Note: (Entire, Regular) is fact. system on Strat:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "1"'] & & Y \\ & Z \arrow[ur, "f"'] & \end{tikzcd}$$

where $Z = X$ as simp. sets & $a \in Z$ thin $\Leftrightarrow$ Fa is thin.

Adjunctions:

$$\begin{tikzcd} \mathrm{Strat} \arrow[r, "u"] & \mathrm{SSet} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd} \qquad L \dashv u \dashv R$$

where L makes only degeneracies thin & R makes all simplices thin.

Complicial horn inclusions

Def^n) Let $0 \le k \le n$. Then $\Delta^k[n]$ denotes the n-simplex $\Delta[n]$ & we

- declare a non-degenerate m-simplex to be thin if contains $\{k-1, k, k+1\} \cap [n]$.

Examples - The n-simplex in $\Delta^k[n]$ is thin.

- All $(n-1)$-simps except $(k-1)$'th, k'th & $(k+1)$'th are thin.

- We consider k-horn $\Lambda^k[n] \hookrightarrow_r \Delta^k[n]$ as a stratified simplicial subset by declaring the inclusion to be regular.

Complicial horn inclusions

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- Below are pictures of the horn inclusions. Those parts in horn are in black, the others in red. Labels are only intended to suggest interpretation.

$\Delta^0(2)$ $\qquad$ $\Delta^1(2)$ $\qquad$ $\Delta^2(2)$

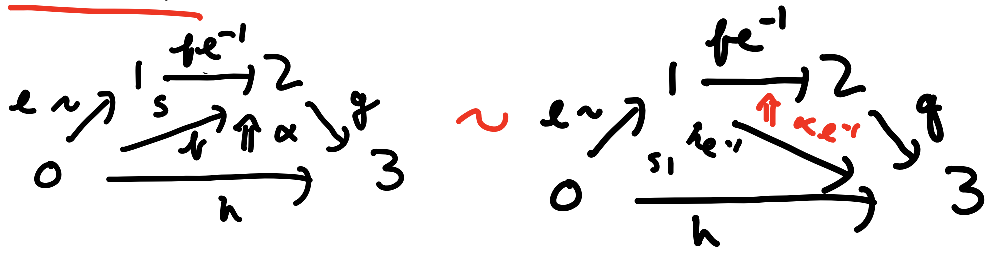
$$
\begin{tikzcd}[column sep=small]
& 1 \arrow[dr, "fe^{-1}"] \arrow[d, phantom, "s_1"] & \\
0 \arrow[ur, "e \simeq"] \arrow[rr, "f"'] & {} & 2
\end{tikzcd}
,\quad
\begin{tikzcd}[column sep=small]
& 1 \arrow[dr, "g"] \arrow[d, phantom, "s_1"] & \\
0 \arrow[ur, "f"] \arrow[rr, "gf"'] & {} & 2
\end{tikzcd}
,\quad
\begin{tikzcd}[column sep=small]
& 1 \arrow[dr, "e \simeq"] \arrow[d, phantom, "s_1"] & \\
0 \arrow[ur, "e^{-1}f"] \arrow[rr, "f"'] & {} & 2
\end{tikzcd}
$$

$\Delta^0(2) : [-1 < 0 < 1] \cap [0 < 1 < 2] = 0 < 1$

$\Delta^1(2)$ - no thin 1-simps

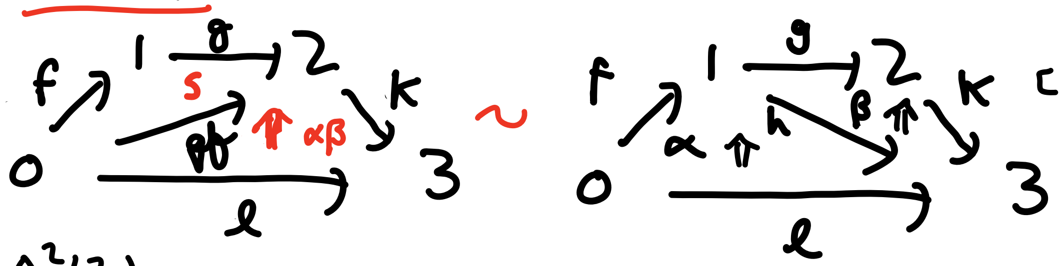
$\Delta^2(2)$ - $1 \to 2$ is thin.

$\Delta^0(3)$



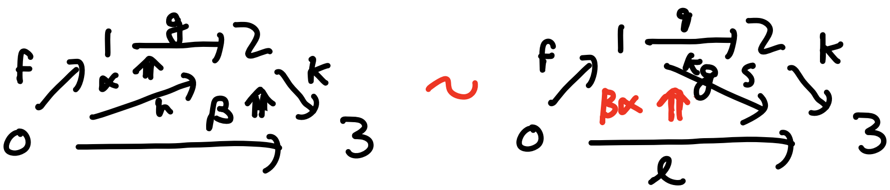
$[-1<0<1]$

$\Delta^1(3)$



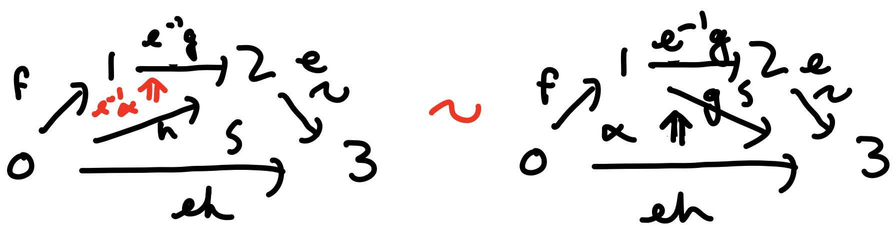
$[0<1<2]$

$\Delta^2(3)$



$[1<2<3]$

$\Delta^3(3)$



$[2<3<4]$

Complicial thinness extensions

- Consider $\Delta^k[n]$.
- Have $\Delta^k[n] \xrightarrow[e]{} \Delta^k[n]' \xhookrightarrow[e]{} \Delta^k[n]''$

  where
  - all have same underlying simplicial set
  - in $\Delta^k[n]'$, declare $(k-1), (k+1)$'th faces thin.
  - in $\Delta^k[n]''$, declare also k'th face thin — so all $(n-1)$-simplices thin.

- Injectivity against $\Delta^k[n]' \xhookrightarrow[e]{} \Delta^k[n]''$ says composite of thin simplices is thin.

- Together, the complicial horn inclusions & complicial thinness extensions are called the elementary (anodyne) extensions.

Def^n) A complicial set is a stratified simplicial set injective wrt to elementary extensions.

(∞,n)-cats

- $X \in Strat$ is <u>n-trivial</u> if all k-simplices for $k>n$ are thin.

- <u>n-trivial complicial sets</u> provide a model for (∞,n)-cats.

  - Adjunction

$$\begin{tikzcd} n\text{-}Strat \arrow[r, hook] & Strat \arrow[l, bend right, "tr_n"'] \arrow[l, bend left, "core_n"] \end{tikzcd}$$

- where tr_n makes all thin for $k>n$,
- $core_n$ restricts to those simplices whose faces above dimension n are all thin.

- The two right adjoints above restrict to complicial sets, giving

$$(\infty,n)\text{-Cats} \underset{\perp}{\overset{\longrightarrow}{\longleftarrow}} (\infty,\infty)\text{-Cats}$$

Street-Roberts conjecture

Def^n) A strict complicial set is a strat. simp. set which is orthogonal to the elementary extensions.

There is a nerve functor

$N : \omega\text{-}\mathrm{Cat} \longrightarrow \mathrm{Strat}$ sending X to its Street nerve, with only identities marked as thin.

Theorem (Verity)

N is fully faithful & has in its essential image exactly the strict complicial sets.

Theorem (Verity)

There is a model structure on Strat whose fibrant objects are the saturated complicial sets: those whose thin simplices are precisely the equivalences.

Remark) Lurie's ∞-bicats are similar, but only considers marked 2-simplices, aka - scaled simplicial sets.

Ref) Emily Riehl: Complicial sets, an overture.