3.2.3. Moment Generating, and Characteristic Functions

The usefulness of moments partly stems from the fact that knowledge of them determines the form of the density function. Formally, if the moments μ'_n of a random variable x exist and the series

$$\sum_{n=1}^{\infty} \frac{\mu'_n}{n!} r^n \tag{3.10}$$

converges absolutely for some $r > 0$, then the set of moments μ'_n uniquely determines the density function. There are exceptions to this statement, but fortunately it is true for all the distributions commonly met in physical science. In practice, knowledge of the first few moments essentially determines the general characteristics of the distribution and so it is worthwhile to construct a method that gives a representation of all the moments. Such a function is called a *moment generating function (mgf)* and is defined by

$$M_x(t) \equiv E[e^{xt}]. \tag{3.11}$$

For a discrete random variable x, this is

$$M_x(t) = \sum e^{xt} f(x), \tag{3.12a}$$

and for a continuous variable,

$$M_x(t) = \int_{-\infty}^{+\infty} e^{xt} f(x)\mathrm{d}x. \tag{3.12b}$$

The moments may be generated from (3.11) by first expanding the exponential,

$$M_x(t) = E\left[1 + xt + \frac{1}{2!}(xt)^2 + \cdots\right] = \sum_{n=0}^{\infty} \frac{1}{n!} \mu'_n t^n,$$

then differentiating n times and setting $t = 0$, that is:

$$\mu'_n = \left.\frac{\partial^n M_x(t)}{\partial t^n}\right|_{t=0}. \tag{3.13}$$

For example, setting $n = 0$ and $n = 1$, gives $\mu'_0 = 1$ and $\mu'_1 = \mu$. Also, since the mgf about any point λ is

$$M_\lambda(t) = E[\exp\{(x - \lambda)t\}],$$

then if $\lambda = \mu$,

$$M_\mu(t) = e^{-\mu t} M_x(t). \tag{3.14}$$

An important use of the mgf is to compare two density functions $f(x)$ and $g(x)$. If two random variables possess mgfs that are equal for some interval symmetric about the origin, then $f(x)$ and $g(x)$ are identical density functions. It is also straightforward to show that the mgf of a sum of independent random variables is equal to the product of their individual mgfs.

It is sometimes convenient to consider, instead of the mgf, its logarithm. The Taylor expansion[3] for this quantity is

$$\ln M_x(t) = \kappa_1 t + \kappa_2 \frac{t^2}{2} + \cdots,$$

where κ_n is the *cumulant* of order n, and

$$\kappa_n = \left.\frac{\partial^n \ln M_x(t)}{\partial t^n}\right|_{t=0}.$$

Cumulants are simply related to the central moments of the distribution, the first few relations being

$$\kappa_i = \mu_i \ (i = 1, 2, 3), \quad \kappa_4 = \mu_4 - 3\mu_2^2 .$$

For some distributions the integral defining the mgf may not exist and in these circumstances the Fourier transform of the density function, defined as

$$\phi_x(t) \equiv E[e^{itx}] = \int_{-\infty}^{+\infty} e^{itx} f(x)\mathrm{d}x = M_x(it), \tag{3.15}$$

may be used. In statistics, $\phi_x(t)$ is called the *characteristic function (cf)*. The density function is then obtainable by the Fourier transform theorem (known in this context as the *inversion theorem*):

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \phi_x(t)\mathrm{d}t. \tag{3.16}$$

The cf obeys theorems analogous to those obeyed by the mgf, that is: (a) if two random variables possess cfs that are equal for some interval symmetric about the origin then they have identical density functions; and (b) the cf of a sum of independent random variables is equal to the product of their individual cfs. The converse of (b) is however untrue.

EXAMPLE 3.5

Find the moment generating function of the density function used in Example 3.2 and calculate the three moments μ_1', μ_2', and μ_3'.

Using definition (3.12b),

$$M_x(t) = \int_0^{\infty} e^{xt} f(x)\mathrm{d}x = \frac{1}{2}\int_0^{\infty} e^{xt} x^2 e^{-x}\mathrm{d}x = \frac{1}{2}\int_0^{\infty} e^{-x(1-t)} x^2 \mathrm{d}x,$$

[3]Some essential mathematics is reviewed briefly in Appendix A.

which integrating by parts gives:

$$M_x(t) = \left\{ -\frac{e^{-x(1-t)}}{2(1-t)^3}\left[(1-t)^2x^2 + 2(1-t)x + 2\right] \right\}_0^\infty = \frac{1}{(1-t)^3}.$$

Then, using (3.13), the first three moments of the distribution are found to be

$$\mu_1' = 3, \quad \mu_2' = 12, \quad \mu_3' = 60.$$

EXAMPLE 3.6

(a) Find the characteristic function of the density function:

$$f(x) = \begin{cases} 2x/a^2 & a \le x < 0 \\ 0 & \text{otherwise} \end{cases},$$

and (b) the density function corresponding to a characteristic function $e^{-|t|}$.

(a) From (3.15),

$$\phi_x(t) = E[e^{itx}] = \frac{2}{a^2}\int_0^a e^{itx}x\mathrm{d}x.$$

Again, integration by parts gives

$$\phi_x(t) = \frac{2}{a^2}\left[\frac{e^{itx}}{(it)^2}(itx - 1)\right]_0^a = -\frac{2}{a^2t^2}\left[e^{ita}(ita - 1) + 1\right].$$

(b) From the inversion theorem,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|t|}e^{-itx}\mathrm{d}x = \frac{1}{\pi}\int_0^\infty e^{-t}\cos(tx)\mathrm{d}x,$$

where the symmetry of the circular functions has been used. The second integral may be evaluated by parts to give

$$\pi f(x) = \left[-e^{-t}\cos(tx)\right]_0^\infty - x\int_0^\infty e^{-t}\sin(tx)\mathrm{d}t$$

$$= 1 - x\left\{\left[-e^{-t}\sin(tx)\right]_0^\infty + x\int_0^\infty e^{-t}\cos(tx)\mathrm{d}t\right\} = 1 - \pi x^2 f(x).$$

Thus,

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty \le x \le \infty.$$

This is the density of the Cauchy distribution that we will meet again in Section 4.5.