Minimum value for central pressure of star

We have only two of the four equations, and no knowledge yet of material composition or physical state. But we can deduce a minimum central pressure.

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \qquad \frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

Divide these two equations:

$$\frac{dP(r)}{dr} \Big/ \frac{dM(r)}{dr} \equiv \frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

Integrations from center to surface gives

$$P_c - P_s = \int_0^{M_s} \frac{GM}{4\pi r^4} dM$$

Lower limit to RHS:

$$\int_0^{M_s} \frac{GM}{4\pi r^4} dM > \int_0^{M_s} \frac{GM}{4\pi r_s^4} dM = \frac{GM_s^2}{8\pi r_s^4}$$

Hence we have

$$P_c - P_s > \frac{GM_s^2}{8\pi r_s^4}$$

We can approximate the pressure at the surface of the star to be zero:

$$P_c > \frac{GM_s^2}{8\pi r_s^4}$$

For example for the Sun:

$P_c = 4.5 \times 10^{13}\ \mathrm{Nm}^{-2} = 4.5 \times 10^{8}$ atmospheres

This seems rather large for gaseous material – we shall see that this is not an ordinary gas.

The Virial theorem

Again lets take the two equations of hydrostatic equilibrium and mass conservation and divide them

$$\frac{dP(r)}{dr} \bigg/ \frac{dM(r)}{dr} \equiv \frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

Now multiply both sides by $4\pi r^2$

$$4\pi r^3 dP = -\frac{GM}{r} dM$$

And integrate over the whole star

$$3\int_{P_c}^{P_s} V dP = -\int_0^{M_s} \frac{GM}{r} dM$$

Where V = vol contained within radius r

Use integration by parts to integrate LHS

$$3[PV]_c^s - 3\int_{V_c}^{V_s} P dV = -\int_0^{M_s} \frac{GM}{r} dM$$

At centre, $V_r = 0$ and at surface $P_s = 0$

The Virial theorem

Hence we have

$$3\int_0^{V_s} PdV + \int_0^{M_s} \frac{GM}{r} dM = 0$$

Now the right hand term is the total gravitational potential energy of the star or it is the energy released in forming the star from its components dispersed to infinity.

Thus we can write the ***Virial Theorem :***

$$3\int_0^{V_s} PdV + \Omega = 0$$

This is of great importance in astrophysics and has many applications. We shall see that it relates the gravitational energy of a star to its thermal energy

Minimum mean temperature of a star

We have seen that pressure P is an important term in the equation of hydrostatic equilibrium and the Virial theorem.

What physical processes give rise to this pressure – which are the most important ?

- Gas pressure P_g
- Radiation pressure P_r
- We shall show that P_r is negligible in stellar interiors and pressure is dominated by P_g

To do this we first need to estimate the minimum mean temperature of a star

Consider the Ω term, which is the gravitational potential energy:

$$-\Omega = \int_0^{M_s} \frac{GM}{r} dM$$

We can obtain a lower bound on the RHS by noting: at all points inside the star $r > r_s$ and hence $1/r > 1/r_s$

$$\Rightarrow \int_0^{M_s} \frac{GM}{r} dM > \int_0^{M_s} \frac{GM}{r_s} dM = \frac{GM_s}{2r_s}$$

Now $dM = \rho dV$ and the Virial theorem can be written

$$-\Omega = 3\int_0^{V_s} P dV = 3\int_0^{M_s} \frac{P}{\rho} dM$$

Now pressure is sum of radiation pressure and gas pressure: $P = P_g + P_r$

Assume, for now, that stars are composed of ideal gas with negligible P_r

$$P = nkT = \frac{k\rho T}{m}$$

The equation of state of ideal gas

k … Boltzmann's constant; m … average mass of particles

n … number of particles per m^3

Hence we have

$$-\Omega = 3\int_0^{M_s} \frac{P}{\rho} dM = 3\int_0^{M_s} \frac{kT}{m} dM$$

And we may use the inequality derived above to write

$$-\Omega = 3\int_0^{M_s} \frac{kT}{m} dM > \frac{GM_s^2}{2r_s}$$

$$\Rightarrow \int_0^{M_s} TdM > \frac{GM_s^2 m}{6kr_s}$$

We can think of the LHS as the sum of the temperatures of all the mass elements dM which make up the star

The mean temperature of the star $\bar{T}$ is then just the integral divided by the total mass of the star M_s

$$\Rightarrow M_s \bar{T} = \int_0^{M_s} TdM$$

$$\bar{T} > \frac{GM_s m}{6kr_s}$$

Minimum mean temperature of the Sun

As an example for the Sun we have:

$$\bar{T} > 4 \times 10^{6} \frac{m}{m_H} \text{ K} \qquad \text{where } m_H = 1.67 \times 10^{-27} \text{ kg}$$

Now we know that Hydrogen is the most abundant element in normal stars and for a fully ionised hydrogen star $m/m_H = {}^1/_2$ (as there are two particles, p + e^-, for each Hydrogen atom). And for any other element m/m_H is greater

$$\bar{T}_{Sun} > 2 \times 10^{6}\text{K}$$

Physical state of stellar material

We can also estimate the mean density of the Sun using:

$$\bar{\rho} = \frac{3M_{Sun}}{4\pi r_{Sun}^3} = 1.4 \times 10^3 \text{ kgm}^{-3}$$

Mean density of the Sun is only a little higher than water and other ordinary liquids. We know such liquids become gaseous at T much lower than $\bar{T}_{Sun}$

Also the average K.E. of particles at $\bar{T}_{Sun}$ is much higher than the ionisation potential of Hydrogen. Thus the gas must be highly ionised, i.e. is a plasma.

It can thus withstand greater compression without deviating from an ideal gas.

Note that an ideal gas demands that the distances between the particles are much greater than their sizes, and nuclear dimension is 10^{-15} m compared to atomic dimension of 10^{-10} m

Lets revisit the issue of radiation versus gas pressure. We assumed that the radiation pressure was negligible. The pressure exerted by photons on the particles in a gas is:

$$P_{rad} = \frac{aT^4}{3}$$

Where a is the radiation density constant

Now compare gas and radiation pressure at a typical point in the Sun

$$\frac{P_r}{P_g} = \frac{aT^4}{3} \Big/ \frac{kT\rho}{m} = \frac{maT^3}{3k\rho}$$

Taking $T \sim \bar{T} = 2\times10^6$ K, $\rho \sim \bar{\rho} = 1.4\times10^3$ kgm^{-3} and $m = \frac{1.67\times10^{-27}}{2}$ kg

Gives $\frac{P_r}{P_g} \sim 10^{-4}$

Hence radiation pressure appears to be negligible at a typical (average) point in the Sun. In summary, with now know of how energy is generated in stars and we have been able to derive a value for the Sun’s internal temperature with negligible radiation pressure.

Mass dependency of radiation to gas pressure

However we shall later see that P_r does become significant in higher mass stars. To give a basic idea of this dependency: replace ρ in the ratio equation above:

$$\frac{P_r}{P_g} = \frac{maT^3}{3k\left(\frac{3M_s}{4\pi r_s^3}\right)} = \frac{4\pi ma}{9k}\frac{r_s^3 T^3}{M_s}$$

$$\text{And from the Virial theorem} \quad : \ \overline{T} \sim \frac{M_s}{r_s}$$

$$\Rightarrow \frac{P_r}{P_g} \propto M_s^2$$

i.e. P_r becomes more significant in higher mass stars.