Unit 9 – Infinity and infinitesimal

1. Can you explain what "infinity" and "infinitesimal" mean?

2. Listening. EXAM PRACTICE. Listen to the recording about the notions of infinity and infinitesimal and fill in the gaps in the summary with suitable words:
http://www.youtube.com/watch?v=jd6MWUl9Rmc

The notions of infinity and infinitesimal are very (1) ……………………………… to confront.

Infinity is presented as being large without (2) ……………………………… and an infinitesimal being so little you can never see it.

Both these concepts are important in (3) ……………………………… and also outside mathematics.

Mathematicians started to understand infinity around (4) ……………………………… .

The speaker mentions two Greek philosophers: Parmenides said that nothing in the (5)…………………………… moves as opposed to the argument that everything moves, which was closer to the (6) ……………………………… .

If we looked at the board which seems not to move, we would see (7) ……………………………… of atoms. Parmenides' pupil, Zeno (Xeno) is known for paradoxes which (8)……………………………… the concept of motion.

3. Reading. The text from http://platonicrealms.com/encyclopedia/Zenos-Paradox-of-the-Tortoise-and-Achilles

Zeno's Paradox of the Tortoise and Achilles
Zeno of Elea (circa 450 b.c.) is credited with creating several famous paradoxes, but by far the best known is the paradox of the Tortoise and Achilles. (Achilles was the great Greek hero of Homer's The Iliad.) It has inspired many writers and thinkers through the ages, notably Lewis Carroll and Douglas Hofstadter, who also wrote dialogues involving the Tortoise and Achilles.

Read the text and order the paragraphs A - I.

1) ___, 2) ___, 3) ___, 4) ___, 5) ___, 6) ___, 7) ___, 8) ___, 9) ___

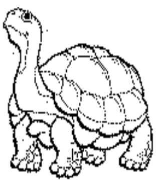
A] "And so you see, in each moment you must be catching up the distance between us, and yet I – at the same time – will be adding a new distance, however small, for you to catch up again." "Indeed, it must be so," said Achilles wearily.

B] “How big a head start do you need?” he asked the Tortoise with a smile. “Ten meters,” the latter replied.
Achilles laughed louder than ever. “You will surely lose, my friend, in that case,” he told the Tortoise, “but let us race, if you wish it.”

C] The Tortoise challenged Achilles to a race, claiming that he would win as long as Achilles gave him a small head start. Achilles laughed at this, for of course he was a mighty warrior and swift of foot, whereas the Tortoise was heavy and slow.

D] “And so you can never catch up,” the Tortoise concluded sympathetically.
“You are right, as always,” said Achilles sadly – and conceded the race.

E] “Suppose,” began the Tortoise, “that you give me a 10-meter head start. Would you say that you could cover that 10 meters between us very quickly?”
“Very quickly,” Achilles affirmed.
“And in that time, how far should I have gone, do you think?”
“Perhaps a meter – no more,” said Achilles after a moment's thought.



F)“Very quickly indeed!”
“And yet, in that time I shall have gone a little way farther, so that now you must catch that distance up, yes?”

G]“On the contrary,” said the Tortoise, “I will win, and I can prove it to you by a simple argument.”
“Go on then,” Achilles replied, with less confidence than he felt before. He knew he was the superior athlete, but he also knew the Tortoise had the sharper wits, and he had lost many a bewildering argument with him before this.

H] “Very well,” replied the Tortoise, “so now there is a meter between us. And you would catch up that distance very quickly?”

I]“Ye-es,” said Achilles slowly.
“And while you are doing so, I shall have gone a little way farther, so that you must then catch up the new distance,” the Tortoise continued smoothly. Achilles said nothing.

4. EXAM PRACTICE. Find words from the text which are synonyms for the expressions below. The words occur in the text in the same order as their explanations.

still existing ……………

result ……………

fact ……………

clearly ……………

mistake ……………

impossible to doubt ……………

Zeno's Paradox may be rephrased as follows. Suppose I wish to cross the room. First, of course, I must cover half the distance. Then, I must cover half the remaining distance. Then, I must cover half the remaining distance. Then I must cover half the remaining distance . . . and so on forever. The consequence is that I can never get to the other side of the room.

What this actually does is to make all motion impossible, for before I can cover half the distance I must cover half of half the distance, and before I can do that I must cover half of half of half of the distance, and so on, so that in reality I can never move any distance at all, because doing so involves moving an infinite number of small intermediate distances first.

Now, since motion obviously is possible, the question arises, what is wrong with Zeno? What is the "flaw in the logic?" If you are giving the matter your full attention, it should begin to make you squirm a bit, for on its face the logic of the situation seems unassailable. You shouldn't be able to cross the room, and the Tortoise should win the race! Yet we know better!

5. Are you able to resolve Zeno's paradox? The following statements can help: decide which of them are Zeno's assumptions and which are the key to explaining the paradox.

…adding up an infinite number of distances should give an infinite distance …

… the sum of infinite segments can add up to a finite amount …

… the Greek philosophers did not have the techniques to solve this problem …

… it takes an infinite amount of time to complete an infinite number of tasks …

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ . \ . \ .$$

6. Based on the previous exercise, write down a conclusion to Zeno's paradox, using suitable linking expressions, e.g. as, since, while, …

7. Do you know any other paradoxes involving the notions of infinity and infinitesimal? Discuss the following paradoxes.

a) THOMPSON'S LAMP: Consider a lamp, with a switch. Hit the switch once, it turns it on. Hit it again, it turns it off. Let us imagine there is a being with supernatural powers who likes to play with this lamp as follows. First, he turns it on. At the end of one minute, he turns it off. At the end of half a minute, he turns it on again. At the end of a quarter of a minute, he turns it off. In one eighth of a minute, he turns it on again. And so on, hitting the switch each time after waiting exactly one-half the time he waited before hitting it the last time. Applying the above discussion, it is easy to see that all these infinitely many time intervals add up to exactly two minutes.

QUESTION 1: At the end of two minutes, is the lamp on, or off?

QUESTION 2: Here the lamp started out being off. Would it have made any difference if it had started out being on?

b) ROSS-LITTLEWOOD PARADOX: Consider an empty vase and an infinite supply of balls and perform an infinite number of steps such that at each step 10 balls are added to the vase and 1 ball is removed from it. To complete an infinite number of steps, it is assumed that the vase is empty at one minute before noon, and that the following steps are performed:

- the first step is performed at 30 seconds before noon,
- the second step is performed at 15 seconds before noon.

Each subsequent step is performed in half the time of the previous step, i.e. step n is performed at 2^{-n} minutes before noon.

QUESTION: How many balls are in the vase when the task is finished?