WEEK 1 : AFFINE GEOMETRY

Let $\mathcal{U}$ be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

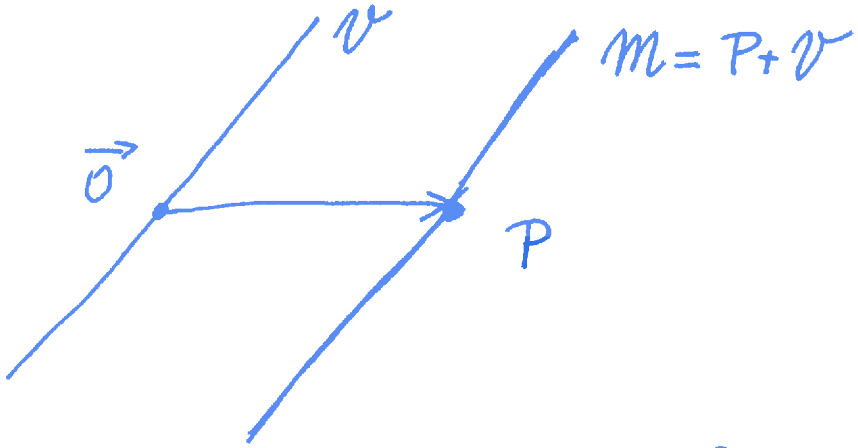
__Definition__ A subset (nonempty) $\mathcal{M} \subseteq \mathcal{U}$ is an __affine subspace__ of $\mathcal{U}$, if it is of the form

$$\mathcal{M} = P + \mathcal{V} = \{P + v \in \mathcal{U},\ v \in \mathcal{V}\}$$

where $P \in \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{U}$ is a vector subspace.

__Remark__ Elements of $\mathcal{M}$ are called points rather than vectors.

__Example 1__ $\mathcal{U} = \mathbb{R}^2$



Affine subspaces in $\mathbb{R}^2$ are:

(1) all points. In this case $\mathcal{V} = \{\vec{0}\}$

(2) lines. In this case $\mathcal{V}$ is a line going through origin.

(3) $\mathbb{R}^2$. $(\mathcal{V} = \mathbb{R}^2)$

__Example 2__ $\mathcal{M} = \{x \in \mathbb{R}^n,\ Ax = b\}$, where

A is a matrix $k \times n$, $b \in \mathbb{R}^k$, such that $h(A) = h(A|b)$ (h stands for rank)

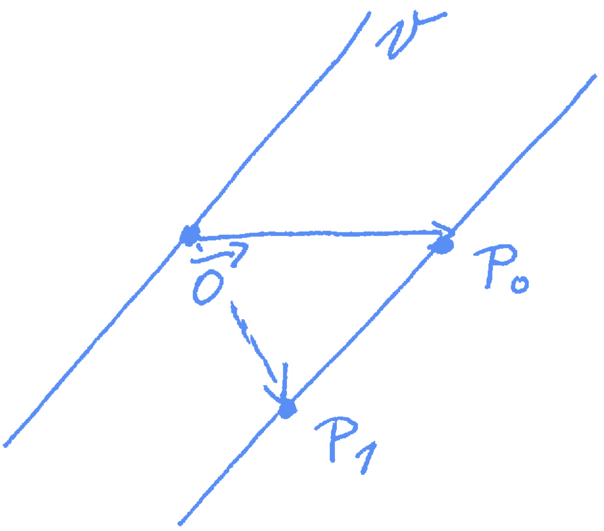
In this case the system of linear equations $Ax = b$ has a solution and $\mathcal{M}$ is not empty.

Let $y \in \mathbb{R}^n$ be a solution of the system $Ax = b$. Then

$$\mathcal{M} = y + \{z \in \mathbb{R}^n;\ Az = 0\}$$

(Theorem: Any solution x of $Ax = b$ is of the form $x = y + z$ where z is a solution of $Az = 0$.)

In the definition $\mathcal{M} = P + V$ the point P is not determined uniquely. See the picture



$\mathcal{M} = P_0 + V = P_1 + V$

Lemma The subspace $\mathcal{V}$ in the definition

$$\mathcal{M} = P + \mathcal{V}$$

is determined uniquely.

Definition Because of lemma we can define direction of $\mathcal{M}$ as

$$Z(\mathcal{M}) = \mathcal{V},$$

and dimension of $\mathcal{M}$ as

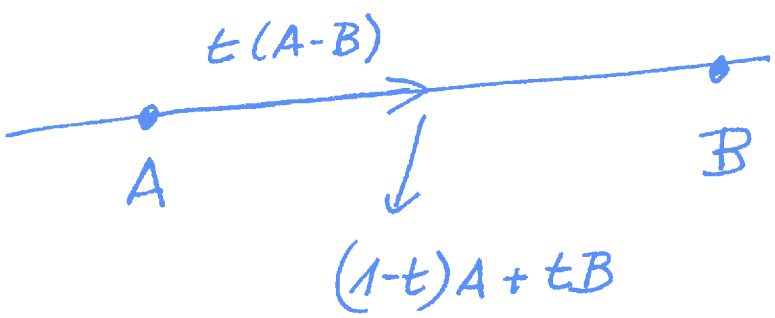
$$\dim \mathcal{M} = \dim \mathcal{V}.$$

Affine combination of points

Let $A \neq B$ be two points in the vector space $\mathcal{U}$. Then the affine combination of point ~~an~~ A and B is a point

$$(1-t)A + tB = A + t(B-A).$$

All affine combinations form a line going through A and B.

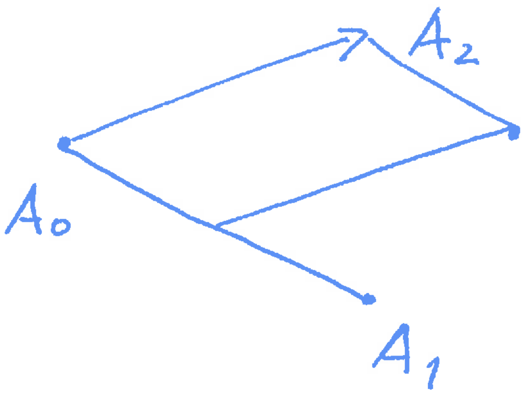


Affine combination of $n+1$ points $A_0, A_1, \dots A_n$ is the point

$$\sum_{i=0}^{n} t_i A_i \quad , \text{ where } \quad \sum_{i=0}^{n} t_i = 1 .$$

We have

$$\sum_{i=0}^{n} t_i A_i = (1-t_1-t_2-\dots-t_n)A_0 + t_1 A_1 + t_2 A_2 + \dots + t_n A_n = A_0 + t_1(A_1-A_0) + t_2(A_2-A_0) + \dots + t_n(A_n-A_0)$$



$$A_0 + \tfrac{1}{2}(A_1-A_0) + 1\cdot(A_2-A_0) = -\tfrac{1}{2}A_0 + \tfrac{1}{2}A_1 + 1\cdot A_2$$

Theorem 1 In every affine subspace $\mathcal{M}$ with any points $A_0, A_1, \dots, A_n \in \mathcal{M}$ their affine combination lies in $\mathcal{M}$ as well.

$$\sum_{i=0}^{n} t_i A_i \in \mathcal{M} \quad \text{for} \quad \sum_{i=0}^{n} t_i = 1 .$$

Proof: $\mathcal{M} = P + \mathcal{V}$, where $\mathcal{V} \leq \mathcal{U}$ is a vector subspace. $A_i = P + v_i$, $v_i \in \mathcal{V}$. Then

$$\sum_{i=0}^{n} t_i A_i = \sum_{i=0}^{n} t_i (P+v_i) = \Big(\sum_{i=0}^{n} t_i\Big) P + \sum_{i=0}^{n} t_i v_i = P + \sum_{i=0}^{n} t_i v_i \in P + \mathcal{V} = \mathcal{M} .$$

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<u>Theorem 2</u> Let $\mathcal{M}$ be a nonemty subset of the vector space $\mathcal{U}$ such that for every $A, B \in \mathcal{M}$ also $(1-t)A + tB \in \mathcal{M}$. Then $\mathcal{M}$ is an affine subspace.

<u>Proof</u> There is a point $P \in \mathcal{M}$. Define

$$\mathcal{V} = \{ A - P \in \mathcal{U}, \; A \in \mathcal{M} \}.$$

Then

$$\mathcal{M} = P + \mathcal{V}.$$

It suffices to show that $\mathcal{V}$ is a vector subspace of $\mathcal{U}$.

Let $A-P, \; B-P \in \mathcal{V}$, where $A, B, P \in \mathcal{M}$.

Then

$$(A-P) + (B-P) = \underbrace{\left\{ 2 \Big(\underbrace{\tfrac{1}{2} A + \tfrac{1}{2} B}_{\in \mathcal{M}} \Big) - 1 \cdot P \right\}}_{\in \mathcal{M}} - P \in \mathcal{V}$$

$$a(A-P) = \underbrace{\{ aA + (1-a)P \}}_{\in \mathcal{M}} - P \in \mathcal{V}.$$

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So we can give <u>an equivalent definition of affine subspace</u> $\mathcal{M}$ as a nonempty subset of the vector space $\mathcal{U}$ such that for every A and B in $\mathcal{M}$, $A \neq B$, also the line $\overleftrightarrow{AB}$ lies in $\mathcal{M}$.

Parametric description of affine subspace

$\mathcal{M} = P + \mathcal{V}$, $\mathcal{V}$ has a basis $v_1, v_2, \dots v_k$, so every point $M \in \mathcal{M}$ can be written in the form

$$M = P + t_1 v_1 + t_2 v_2 + \dots + t_k v_k .$$

Examples

1. Parametric description of a line in $\mathbb{R}^3$ is

$$p : \quad M = P + t\vec{v}$$

2. Parametric description of a plane in $\mathbb{R}^3$ is

$$\rho : \quad M = P + t\vec{u} + s\vec{v} .$$

Implicit description of affine space in $\mathbb{K}^n$

$$\mathcal{M} = \{ x \in \mathbb{K}^n , \ Ax = b \}$$

where A is a matrix $k \times n$ and $b \in \mathbb{K}^k$ for suitable k.

Examples

1. A plane in $\mathbb{R}^3$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b$$

where $(a_{11}, a_{12}, a_{13}) \neq (0,0,0)$.

2 A line in $\mathbb{R}^3$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$h\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = 2 = h\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \end{array}\right)$$

$$\dim M = \dim Z(M) = \dim \{x \in \mathbb{R}^3, Ax = 0\}$$
$$= 3 - h(A) = 3 - 2 = 1.$$

Transition from implicit description to ~~[illegible]~~ parametric description

is easy. It is sufficient to solve the system of equation using parametres.

Suppose $Ax = b$ has a solution

$x_1 = 2 + 3t - s$, $x_2 = 3 - t + 9s$, $x_3 = 1 - 2s$, $x_4 = 2 + t$.

Then

$$X = (2+3t-s,\ 3-t+9s,\ 1-2s,\ 2+t) =$$
$$= \underset{\overset{\|}{P}}{(2,3,1,2)} + \underset{t\,v_1}{t(3,-1,0,1)} + \underset{s\,v_2}{s(-1,9,-2,0)}$$

From parametric description to implicit description

Let $x = P + t_1 v_1 + \dots + t_k v_k$ $\quad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$ $v_i = \begin{pmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{ni} \end{pmatrix}$

In coordinates:

$$x_1 = p_1 + c_{11} t_1 + c_{12} t_2 + \dots + c_{1k} t_k$$
$$\dots\dots\dots\dots\dots\dots$$
$$x_n = p_n + c_{n1} t_1 + c_{n2} t_2 + \dots + c_{nk} t_k$$

This can be written in the matrix form as

$$E x = C t + p \qquad t = \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \; x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Unit matrix

Algorithm:

$$(E \mid C \mid p) \xrightarrow[\text{operations}]{\text{making row}} \left(\begin{array}{c|c|c} A_1 & C_1 & b_1 \\ \hline A & 0 & b \end{array} \right)$$

in such a way that C_1 is in echelon form without zero row.

Then x and satisfies equations

$$A_1 x = C_1 t + b_1$$
$$A x = b$$

The second equation is that we have looked for.

Intersection of affine subspaces $\mathcal{M}$ and $\mathcal{N}$ is again an affine subspace if $\mathcal{M} \cap \mathcal{N} \neq \emptyset$.

In exercises we meet three posibilities:

(1) Both sub spaces are given by a system of equations

$$\mathcal{M} : \quad Ax = b$$
$$\mathcal{N} : \quad Cy = d$$

Then $\mathcal{M} \cap \mathcal{N}$: $\begin{matrix} Az = b \\ Cz = d \end{matrix}$

(2) $\mathcal{M}$ has a parametric description

$$\mathcal{M} : X = P + t_1 v_1 + \dots + t_k v_k$$

$\mathcal{N}$ given by a system of equations

$$\mathcal{N} : \quad Ax = b$$

For x in the system $Ax = b$ we substitute

$$\begin{aligned} x_1 &= p_1 + t_1 c_{11} + \dots + t_k c_{1k} \\ x_2 &= p_2 + \dots \\ &- - - - - - - - - - - \\ x_n &= p_n + t_1 c_{n1} + \dots \, t_k c_{nk} \, . \end{aligned}$$

So we get a system of equations for $t_1, t_2, \dots, t_k$.

(3) Both $\mathcal{M}$ a $\mathcal{N}$ are given by parametric description:

$$\mathcal{M}: \quad P + t_1u_1 + t_2u_2 + t_3u_3$$
$$\mathcal{N}: \quad Q + s_1v_1 + s_2v_2$$

Then

$$\mathcal{M} \cap \mathcal{N} = \{ X = P + t_1u_1 + t_2u_2 + t_3u_3 = Q + s_1v_1 + s_2v_2$$

We solve the system in unknowns t_1, t_2, t_3, s_1, s_2 :

$$t_1u_1 + t_2u_2 + t_3u_3 - s_1v_1 - s_2v_2 = Q - P$$

If we get solution

$$s_2 = \alpha$$
$$s_1 = 2 + \alpha$$

then

$$\mathcal{M} \cap \mathcal{N} = \{ Q + (2+\alpha)v_1 + \alpha v_2 \}$$
$$= \{ Q + 2v_1 + \alpha(v_1 + v_2) \}$$
$$= \underbrace{(Q + 2v_1)}_{\text{point}} + \underbrace{[v_1 + v_2]}_{\text{subspace}}$$

Connection of two affine spaces $\mathcal{M}$ and $\mathcal{N}$ is denoted by

$$\mathcal{M} \sqcup \mathcal{N}$$

and it is the smallest affine subspace containing both $\mathcal{M}$ and $\mathcal{N}$.

If $\mathcal{M}: P + \mathcal{V}$ and $\mathcal{N}: Q + \mathcal{W}$, then

$$\mathcal{M} \sqcup \mathcal{N} : \quad P + [Q-P] + \mathcal{V} + \mathcal{W}$$

where $[Q-P]$ is a linear envelope of the vector $Q-P$ and

$$[Q-P]+\mathcal{V}+\mathcal{W}$$

is the sum of the subspaces $[Q-P]$, $\mathcal{V}$, $\mathcal{W}$ and so it is a vector subspace.

Relative position of two subspaces $\mathcal{M}$ and $\mathcal{N}$ is

(1) $\mathcal{M} \subseteq \mathcal{N}$ or $\mathcal{N} \subseteq \mathcal{M}$

(2) Parallel if $\mathcal{M} \cap \mathcal{N} = \emptyset$, but

$$Z(\mathcal{M}) \subseteq Z(\mathcal{N}) \text{ or } Z(\mathcal{N}) \subseteq Z(\mathcal{M})$$

(3) Intersecting if $\mathcal{M} \cap \mathcal{N} \neq \emptyset$, but

$$Z(\mathcal{M}) \nsubseteq Z(\mathcal{N}) \text{ and } Z(\mathcal{N}) \nsubseteq Z(\mathcal{M})$$

(4) Skew if $\mathcal{M} \cap \mathcal{N} = \emptyset$ and

$$Z(\mathcal{M}) \nsubseteq Z(\mathcal{N}) \text{ and } Z(\mathcal{N}) \nsubseteq Z(\mathcal{M}).$$