The matrix of f in the standard basis $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ is

$$\begin{pmatrix} 1 & 1 & -4 \\ 1 & 0 & 0 \\ -4 & 0 & 2 \end{pmatrix}$$

<u>Theorem</u> There is one-to-one correspondence between quadratic forms on V and symmetric bilinear forms on V.

<u>Proof</u>: f sym. bilinear fom, then

$$q(u) = f(u,u)$$

is a quadratic form according to the definition.

If q is a quadratic form, then

$$f(u,v) = \frac{1}{2}\left(q(u+v) - q(u) - q(v)\right)$$

is the corresponding sym. bil. form.

If $q(u) = f(u,u)$, then

$$q(u+v) - q(u) - q(v) = f(u+v,u+v) - f(u,u) - f(v,v) = f(u,u) + f(u,v) + f(v,u) + f(v,v) - f(u,u) - f(v,v) = 2f(u,v).$$

$\blacksquare$

POLAR BASIS FOR QUADRATIC FORM

is a basis $\mathcal{B}$ such that in its coordinates the qudratic form q can be expressed as

$$q(u) = b_{11} y_1^2 + b_{22} y_2^2 + \dots + b_{nn} y_2^2$$

where $y_1, y_2, \dots y_n$ are coordinates of u in the basis $\mathcal{B}$.

In other words it means that the matrix of corresponding sym. bilinear form is diagonal.

$$B = \begin{pmatrix} b_{11} & & & 0 \\ & b_{22} & & \\ & & \ddots & \\ 0 & & & b_{mn} \end{pmatrix}$$

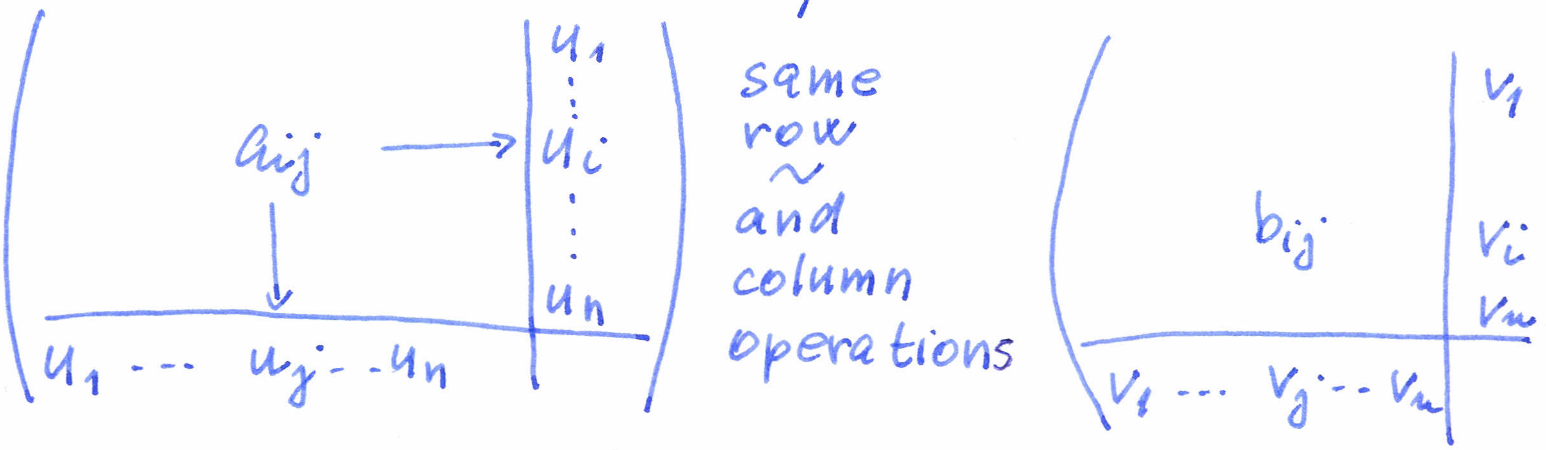
<u>Theorem</u> To every quadratic form (symmetric bilinear form) there is a polar basis. (In fact, there are infinitely many bases.)

<u>Algorithm for finding a polar basis</u>

Quadratic form $q : U \to \mathbb{K}$

Corresponding sym. bilinear form $f : U \times U \to \mathbb{K}$ has the matrix A in the basis $\alpha = (u_1, u_2, \dots u_n)$.

$$A = \left(a_{ij} = f(u_i, u_j) \right)$$



$B = (b_{ij})$ is a new matrix of f in the basis $\beta = (v_1, v_2, \dots v_n)$. The operations can be carried out in such a way that $B = (b_{ij})$ is diagonal.

Apply the algorithm on the quadratic form $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$g(x) = 4x_1x_2 + 6x_1x_3 - 8x_2x_3$$

$$f(x,y) = 2x_1y_2 + 2x_2y_1 + 3x_1y_3 + 3x_2y_1 - 4x_2y_3 - 4x_3y_2$$

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 0 \end{pmatrix} \qquad \alpha = \left(e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & e_1 \\ 2 & 0 & -4 & e_2 \\ 3 & -4 & 0 & e_3 \\ \hline e_1 & e_2 & e_3 & \end{array}\right)$$

add 2-nd row to the 1-st

$$\sim \left(\begin{array}{ccc|c} 2 & 2 & -1 & e_1+e_2 \\ 2 & 0 & -4 & e_2 \\ 3 & -4 & 0 & e_3 \\ \hline e_1 & e_2 & e_3 & \end{array}\right)$$

add the 2-nd column to the 1-st one

$$\sim \left(\begin{array}{ccc|c} 4 & 2 & -1 & e_1+e_2 \\ 2 & 0 & -4 & e_2 \\ -1 & -4 & 0 & e_3 \\ \hline e_1+e_2 & e_2 & e_3 & \end{array}\right)$$

$\sim$ proceed

Homework:

Find a polar basis for the symmetric bilinear form $f : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$.

$$f(x,y) = 2x_1y_2 + 8x_1y_3 + 2x_2y_1 - 2x_2y_3 - 8x_2y_4 + 8x_3y_1 - 2x_3y_2 + 8x_3y_4 - 8x_4y_2 + 8x_4y_3$$