LA- WEEK 8 ORTHOGONAL AND UNITARY OPERATORS, II

Eigenvalues and eigenvectors of unitary and orthogonal operators

Theorem Let $\varphi : U \to U$ be a unitary or orthogonal operator.

(1) All eigenvalues of φ have absolute value equal to 1.

(2) Eigevectors to different eigenvalues are mutually orthogonal.

Proof: (1) Let $\varphi(u) = \lambda u$, $u \neq \vec{0}$. Then

$$\lambda\bar{\lambda}\langle u,u\rangle = \langle \lambda u, \lambda u\rangle = \langle \varphi(u), \varphi(u)\rangle = \langle u,u\rangle$$

Hence $(\lambda\bar{\lambda} - 1)\langle u,u\rangle = 0$

and since $\langle u,u\rangle = \|u\|^2 \neq 0$, we get that

$$|\lambda|^2 = \lambda\bar{\lambda} = 1$$

$(\lambda = a+ib,\ a, b \in \mathbb{R},\ \bar{\lambda} = a-ib,\ \lambda\bar{\lambda} = a^2+b^2 = |\lambda|^2)$.

(2) Let $\varphi(u_1) = \lambda_1 u_1$, $\varphi(u_2) = \lambda_2 u_2$, $u_1 \neq \vec{0}$, $u_2 \neq \vec{0}$ and $\lambda_1 \neq \lambda_2$. Then

$$\langle u_1, u_2\rangle = \langle \varphi(u_1), \varphi(u_2)\rangle = \langle \lambda_1 u_1, \lambda_2 u_2\rangle = \lambda_1\bar{\lambda}_2\langle u_1, u_2\rangle$$

Hence $(\lambda_1\bar{\lambda}_2 - 1)\langle u_1, u_2\rangle = 0$.

Since $\lambda_1 \neq \lambda_2$, $\lambda_1\lambda_2^{-1} \neq 1$, $\lambda_1\lambda_2^{-1} = \lambda_1\bar{\lambda}_2 \neq 1$, we get $\langle u_1, u_2\rangle = 0$.

Basic theorem on unitary operators

Let $\varphi : U \to U$ be a unitary operator. Then in U there is an orthonormal basis $\alpha = (u_1, u_2, \dots, u_n)$ formed by eigenvectors of φ. In this basis

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} \lambda_1 & 0 & 0 & & 0 \\ 0 & \lambda_2 & 0 & & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigenvalues of φ.

The proof is by induction with respect to $\dim U$. I will not do it here. See pages 7 and 8 in Czech lectures in IS.

Caution The situation for orthogonal operators is much more complicated. We will examine it in the rest of these notes.

Invariant subspaces of orthogonal operators

We will consider orthogonal operator $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(x) = Ax$, where A is an orthogonal matrix. φ can be extended to the operator $\varphi^{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$, $\varphi^{\mathbb{C}}(x) = Ax$ (but here $x \in \mathbb{C}^n$). $\varphi^{\mathbb{C}}$ is a unitary operator since its matrix is unitary

$$A^{-1} = A^T = \bar{A}^T \quad (A \text{ is real!})$$

- Suppose that $\varphi^{\mathbb{C}}$ has in $\mathbb{C}$ an eigenvalue $\lambda = a + ib$, where $b \neq 0$. Since $|a+ib| = 1$, we can write

$$\lambda = a + ib = \cos\varkappa + i\sin\varkappa$$

where $\varkappa \neq k\pi$. Then $\varphi^{\mathbb{C}}$ has another eigenvalue $\bar{\lambda} = a - ib = \cos\varkappa - i\sin\varkappa$.

<u>Proof</u>: Let $u \in \mathbb{C}^n$ be an eigenvector for λ. Then $u = u_1 + iu_2$, $u_1, u_2 \in \mathbb{R}^n$. Write $\bar{u} = u_1 - iu_2$. Then

$$Au = \lambda u$$
$$\overline{Au} = \overline{\lambda u}$$
$$\bar{A}\bar{u} = \bar{\lambda}\bar{u}$$

Since A is a real matrix $\bar{A} = A$ and hence

$$A\bar{u} = \bar{\lambda}\bar{u}$$

and so $\bar{u} = u_1 - iu_2$ is an eigenvector for the eigenvalue $\bar{\lambda} = a - ib$.

Lemma For the eigenvector $u = u_1 + i u_2$ to the eigenvalue $\lambda = a + ib$ it holds

(1) $\|u_1\| = \|u_2\|$, $\langle u_1, u_2 \rangle = 0$.

(2) Two dimensionae subspace $[u_1, u_2] \leq \mathbb{R}^n$ is an invariant with respect to φ. Moreover, φ acts on it as a rotation by the angle φ in the directiom from u_2 to u_1.

Proof (1) Since $\lambda = \overline{\lambda}$, the eigenvectors $u = u_1 + i u_2$ and $\overline{u} = u_1 - i u_2$ are mutually perpendicular.

$$0 = \langle u, \overline{u} \rangle = \langle u_1 + i u_2, u_1 - i u_2 \rangle = \langle u_1, u_1 \rangle + \langle i u_2, u_1 \rangle$$
$$- \langle u_1, i u_2 \rangle - \langle i u_2, i u_2 \rangle = (\langle u_1, u_2 \rangle - (i)(-i) \langle u_2, u_2 \rangle)$$
$$+ i \langle u_2, u_1 \rangle + i \langle u_1, u_2 \rangle = (\|u_1\|^2 - \|u_2\|^2)$$
$$+ 2i \langle u_1, u_2 \rangle$$

Comparing real and imaginary parts we get

$$\|u_1\|^2 - \|u_2\|^2 = 0$$
$$\langle u_1, u_2 \rangle = 0.$$

(2) We have $A(u_1 + i u_2) = (a + ib)(u_1 + i u_2)$

$$A u_1 + i A u_2 = (a u_1 - b u_2) + i(b u_1 + a u_2)$$

Comparing real and imaginary parts we get

$$A u_1 = a u_1 - b u_2$$
$$A u_2 = b u_1 + a u_2 .$$

Hence $[u_1, u_2]$ is an invariant subspace for φ. Let us consider the basis

$$\alpha = (u_2, u_1)$$ (the order is important!)

In this basis

$$\left(\varphi / [u_1, u_2] \right)_{\alpha,\alpha} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\varkappa & -\sin\varkappa \\ \sin\varkappa & \cos\varkappa \end{pmatrix}$$

So φ in $[u_1, u_2]$ is a rotation by the angle $\varkappa$ from u_2 to u_1.

Basic theorem on orthogonal operators

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal operator. Then U is a direct sum of mutually perpendicular subspaces

$$V_1 \oplus V_2 \oplus \dots \oplus V_k$$

of dimensions 1 or 2.

Subspaces V_i of dimension 1 correspond to eigenvalues 1 and -1 and φ acts on them by multiplication by 1 or -1.

Subspaces V_i of dimension 2 correspond to complex eigenvalues $\cos\psi \pm i\sin\psi$, $\psi \neq k\pi$. φ acts on them as a rotation by the angle ψ.

Applications in $\mathbb{R}^2$

$\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x) = Ax$, A is an orthogonal matrix 2×2.

There are the following possibilities

(A) Eigenvalues of A are $1, 1$

Then $A = E$ and $\varphi(x) = x$.

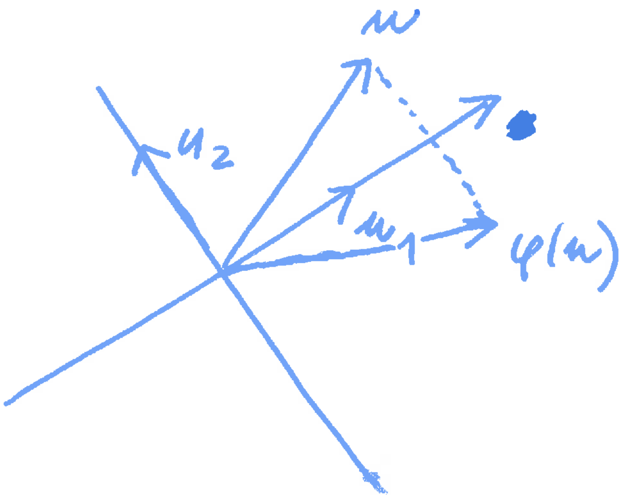
(B) Eigenvalues of A are $-1, -1$.

Then $A = -E$ and $\varphi(x) = -x$ (symmetry with respect the origin).

(C) Eigenvalues of A are $1, -1$ with eigenvectors u_1 and u_2. Since $\varphi(u_1) = u_1$, $\varphi(u_2) = -u_2$, for the basis $\alpha = (u_1, u_2)$

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ and φ is a

reflexion with respect to the ~~line~~ axe given by $[u_1]$.



D) A has eigenvalues $\cos x \pm i \sin x$, $x \in (0, \pi)$.
Then φ is a rotation by the angle x in the direction from u_2 to u_1, where $u_1 + i u_2$ is an ~~eigenvalues~~ eigenvector for $\cos x + i \sin x$.

Applications in $\mathbb{R}^3$

Every orthogonal operator in dimension 3 has at least one eigenvalue ± 1. (Characteristic polynomial has a real root and it has absolute value 1.) In $\mathbb{R}^3$ we can always find an orthonormal basis $\alpha = (u_1, u_2, u_3)$ in which

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{pmatrix}$$

u_1 is an eigenvector to ± 1 and

$u_3 + i u_2$

is an eigenvector to $\cos\psi + i \sin\psi$.

Geometrically, A if the first eigenvalue is 1, φ is the rotation around the axis $[u_1]$ by the angle ψ.

B if the first eigenvalue is -1, φ is a composition of the rotation described above and the reflexion with respect to the plane $[u_2, u_3]$ which is perpendicular to $[u_1]$.

<u>Exercise 1</u> Find which geometric map is described by the operator $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x) = Ax$, where $A = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

<u>Exercise 2</u> Find which geometric transformation is described by the operator $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(x) = Ax$, where $A = \frac{1}{3}\begin{pmatrix} -2 & 1 & -2 \\ -2 & -2 & 1 \\ 1 & -2 & -2 \end{pmatrix}$