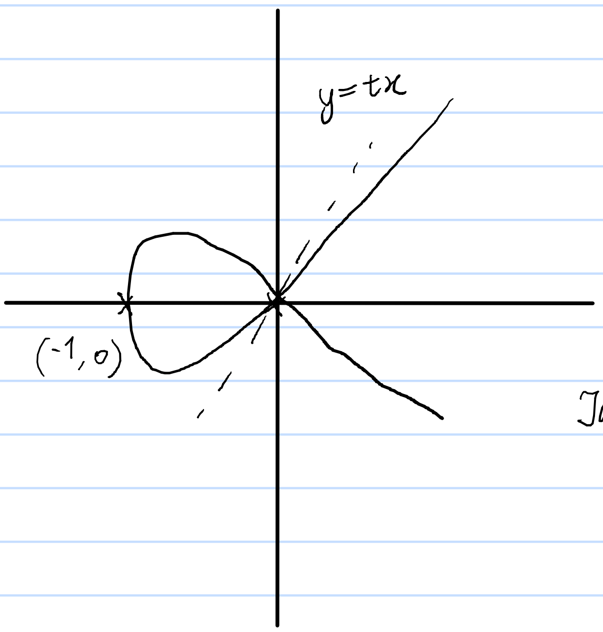
① Rational solutions of curves

i) Find rational solutions to

$$x^2 + x^3 - y^2 = 0$$

Philosophy: pick 1 rational pt on curve, if there is another, the 2 are connected by a straight line with rational slope t.

$(0,0)$ is on this curve



Take $y = tx$, (parametrising)

$$x^2 + x^3 - t^2x^2 = 0$$

$$x^2(1 + x - t^2) = 0$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases}, \quad \begin{cases} x = t^2 - 1 \\ y = t(t^2-1) = t^3 - t \end{cases}$$

By taking $t \in \mathbb{Q}$, we obtain all rational solutions

y^2 : vertical symmetry
x^2 :
x^3 :

Fermat's Last Thm: $x^n + y^n - z^n = 0$

$$\left(\frac{x}{z}\right)^n + \left(\frac{y}{z}\right)^n - 1 = 0 \quad , \frac{x}{z}, \frac{y}{z} \in \mathbb{Q}$$

ii)

In lecture, we know about $x^2 + y^2 - 1 = 0$

~~How~~ about $x^n + y^n - 1 = 0$? $n \geq 3$

We shall see that we cannot perform parametrisation we did in lecture to find all rational solutions.

Suppose to the contrary that we can,

i.e., $x(t) = \frac{p(t)}{r(t)}$, $y(t) = \frac{q(t)}{r(t)}$, here both has $r(t)$ as denominator since $y^n = 1 - x^n$

where $\gcd(p, q, r) = 1$, $p, q, r \in \mathbb{Q}[t]$

Substituting, we have

$$p^n + q^n - r^n = 0 \qquad n \neq 0$$

Differentiating,

$$\cancel{n} p^{n-1} p' + \cancel{n} q^{n-1} q' - \cancel{n} r^{n-1} r' = 0$$

Thus we have

$$\begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix} \begin{pmatrix} p^{n-1} \\ q^{n-1} \\ -r^{n-1} \end{pmatrix} = 0$$

One can check that

$$\begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix} \begin{pmatrix} qr' - rq' \\ rp' - pr' \\ pq' - qp' \end{pmatrix} = 0$$

Note that this is not a trivial solution, since $y'(t) = \left(\frac{q}{r}\right)' \neq 0$

By Rank - Nullity Thm,

the solution space for $\begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$ is of dim 1

$$\Rightarrow \begin{pmatrix} qr' - rq' \\ rp' - pr' \\ pq' - qp' \end{pmatrix} = h \begin{pmatrix} p^{n-1} \\ q^{n-1} \\ -r^{n-1} \end{pmatrix}, \quad h \in \mathbb{Q}(t)$$

Now since $\gcd(p^{n-1}, q^{n-1}, -r^{n-1}) = 1$,

if $h = \frac{f}{g}$, $f, g \in \mathbb{Q}[t]$, $\gcd(f,g) = 1$,

then $\frac{f}{g}\begin{pmatrix} p^{n-1} \\ q^{n-1} \\ -r^{n-1} \end{pmatrix} = \begin{pmatrix} \text{poly} \\ \text{poly} \\ \text{poly} \end{pmatrix}$, so $g = 1$, hence $h \in \mathbb{Q}[t]$

Now,

$$\begin{aligned} \deg qr' = \deg rq' &= \deg q + \deg r - 1 \\ &\geq \deg(qr' - rq') \\ &\geq \deg p^{n-1} \\ &= (n-1)\deg p \end{aligned}$$

$\because hp^{n-1} = qr' - rq'$

Similarly, $\deg r + \deg p - 1 \geq (n-1)\deg q$

$\deg p + \deg q - 1 \geq (n-1)\deg r$

Summing the 3 inequalities, we have

$$2(\deg p + \deg q + \deg r) - 3 \geq (n-1)(\deg p + \deg q + \deg r)$$

$$(3-n)(\deg p + \deg q + \deg r) \geq 3$$

but we assume $n \geq 3$, contradiction

(2) Resultants & Discriminants

Consider $k[x]$, $\quad = (-1)^{mn}\ \mathrm{Res}\,(k,h)$
$\quad = (-1)^{mn} \prod_\delta \prod_\gamma (\delta - \gamma)$
$\quad = (-1)^{mn} \prod_\delta h(\delta)$

Recall:

for $f = a_r x^r + \dots + a_1 x + a_0$

$g = b_s x^s + \dots + b_1 x + b_0$

the Sylvester matrix is defined as

$$\mathrm{Syl}(f,g) = \begin{pmatrix} a_r & & 0 & b_s & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ a_0 & & a_r & b_0 & & b_s \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & a_0 & 0 & & b_0 \end{pmatrix}$$

$(r+s)\times(r+s)$ matrix

and the resultant is defined as

$$\mathrm{Res}(f,g) = \det \mathrm{Syl}(f,g)$$

Next, we can write

$$f = a_r (x-\alpha_1)\cdots(x-\alpha_r) = a_r \prod_i (x-\alpha_i)$$

where α are the roots of f

The discriminant of a poly f is defined as

$$\mathrm{Disc}(f) = (-1)^{\frac{r(r-1)}{2}}\, a_r^{2r-2} \prod_{i\neq j} (\alpha_i - \alpha_j)$$

Recall:

In lectures, we learnt that $\quad f' = r a_r x^{r-1} + \dots + a_1$

Thm. $\mathrm{Res}(f,g) = a_r^s b_s^r \prod_{i,j} (\alpha_i - \beta_j)$, β are roots of g

1) Show $\mathrm{Disc}(f) = \dfrac{1}{(-1)^{\frac{r(r-1)}{2}} a_r} \mathrm{Res}(f,f')$

pf. $f = a_r (x-\alpha_1)\cdots(x-\alpha_r)$

Product rule: $f' = a_r \sum_j \prod_{i\neq j} (x-\alpha_i)$

Now for monic h, k

with roots γ, δ resp., $\quad h(x) = \prod_\gamma (x-\gamma)$, $k(x) = \prod_\delta (x-\delta)$

~~we know~~ $\mathrm{Res}(h,k)$ $\quad = \det$ exchanging columns gives -1

~~we obtain~~ $= \prod_\gamma \prod_\delta (\gamma - \delta)$

$\cancel{\Rightarrow} = \prod_\gamma k(\gamma)$ $\quad \leftarrow$ no need

Now, $\mathrm{Res}(f,f') \overset{(*)}{=} r^r a_r^{2r-1}\, \mathrm{Res}\left(\frac{f}{a_r}, \frac{f'}{r a_r}\right)$

$= r^r a_r^{2r-1} \prod_\ell \frac{1}{r a_r} f'(\alpha_\ell)$ where α_ℓ are roots of f

$= a_r^{r-1} \prod_\ell f'(\alpha_\ell)$

and $f'(\alpha_\ell) = a_r \sum_j \prod_{i\neq j} (\alpha_\ell - \alpha_i)$ $\quad = 0$ if $i = \ell \Rightarrow$ remains $j=\ell$

$= a_r \prod_{i\neq \ell} (\alpha_\ell - \alpha_i)$

So $\mathrm{Res}(f,f') = a_r^{r-1} \prod_\ell a_r \prod_{i\neq \ell} (\alpha_\ell - \alpha_i)$

$= a_r^{2r-1} \prod_{i\neq \ell} (\alpha_\ell - \alpha_i)$

(*)

$$\operatorname{Res}(f,f') = \det \begin{pmatrix} a_r & & & r a_r & & \\ \vdots & \ddots & & \vdots & \ddots & \\ \vdots & & & a_1 & & \\ a_0 & & & & & \end{pmatrix}$$

$$\begin{array}{cc} \underbrace{\qquad\qquad}_{r-1} & \underbrace{\qquad\qquad}_{r} \end{array}$$

$$\operatorname{Res}\left(\frac{f}{a_r}, \frac{f'}{r a_r}\right) = \det \begin{pmatrix} 1 & & & 1 & & \\ \vdots & \ddots & & \vdots & \ddots & \\ \vdots & & & \frac{a_1}{r a_r} & & \\ \frac{a_0}{a_r} & & & & & \end{pmatrix}$$

$$\Rightarrow \operatorname{Res}(f,f') = a_r^{r-1} (r a_r)^r \operatorname{Res}\left(\frac{f}{a_r}, \frac{f'}{r a_r}\right)$$

$$= r^r a_r^{2r-1} \operatorname{Res}\left(\frac{f}{a_r}, \frac{f'}{r a_r}\right)$$

i) Disc for quadratic poly:

$$f = ax^2 + bx + c$$

$$\mathrm{Disc}(f) = \frac{(-1)}{a} \mathrm{Res}(f, f')$$

$$= \frac{-1}{a} \det \begin{pmatrix} a & 2a & 0 \\ b & b & 2a \\ c & 0 & b \end{pmatrix}$$

$$= \frac{-1}{a} a(-b^2 + 4ac)$$

$$= b^2 - 4ac$$

ii) Disc for $f = x^3 + px + q$

$$\mathrm{Disc}(f) = \frac{-1}{1} \mathrm{Res}(f, f')$$

$$= -\det \begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ p & 0 & p & 0 & 3 \\ q & p & 0 & p & 0 \\ 0 & q & 0 & 0 & p \end{pmatrix}$$

$$= -4p^3 - 27q^2$$

iii) Disc for $f = x^2 + 2xy^2 + y + 1 \in \mathbb{C}[y][x]$

Interpret the roots of this discriminant

$$\mathrm{Disc}(f) = \frac{-1}{1} \mathrm{Res}(f, f')$$

$f' = 2x + 2y^2$

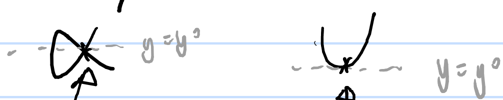
$$= -\det \begin{pmatrix} 1 & 2 & 0 \\ 2y^2 & 2y^2 & 2 \\ y+1 & 0 & 2y^2 \end{pmatrix}$$

$$= -4(-y^4 + y + 1)$$

$\mathrm{Disc}(f) = 0 \Leftrightarrow f$ has multiple roots

So at $y = y_0$ where $-4(-y_0^4 + y_0 + 1)$, the curve $x^2 + 2xy^2 + y + 1 = 0$ has multiple roots

e.g.



multiplicity > 1

mult > 1

∴ f' also has this as root

③ Resultants & Common factors

Consider $\mathbb{k}[x_1, \dots, x_n]$

We may view $\mathbb{k}[x_1, \dots, x_n] \cong \mathbb{k}[x_1, \dots x_{i-1}, x_{i+1}, \dots, x_n][x_i]$

Recall:

In lecture, we learnt

Thm. Non-constant poly $f, g \in \mathbb{k}[x_1, \dots, x_n]$ have common factor in variable $x_i \iff \mathrm{Res}(f, g; x_i) = 0$

i) Solve $\begin{cases} x^2 + y^2 - 4 = 0 \\ 16x^2 + y^2 - 16 = 0 \end{cases}$

i.e., find all common roots of $f = x^2 + y^2 - 4$ and $g = 16x^2 + y^2 - 16$

Algo: for a fixed $y = y_0$, find the common factors of $f(-, y_0), g(-, y_0)$

or for a fixed $x = x_0$, find the common factors of $f(x_0, -), g(x_0, -)$

Compute $\mathrm{Res}(f, g; x)$

$$= \det \begin{pmatrix} 1 & 0 & 16 & 0 \\ 0 & 1 & 0 & 16 \\ y^2-4 & 0 & y^2-16 & 0 \\ 0 & y^2-4 & 0 & y^2-16 \end{pmatrix}$$

$$= 9(5y^2 - 16)^2$$

So if we have y_0 s.t. $9(5y_0^2 - 16)^2 = 0$, then $f(-, y_0)$ and $g(-, y_0)$ have common roots (x_0, y_0), where

$$y_0 = \pm \frac{4}{\sqrt{5}}$$

$$x_0 = \pm \frac{2}{\sqrt{5}}$$

Plugging in y_0:

f: $x^2 + \frac{16}{5} - \frac{20}{5} = 0$

$x^2 - \frac{4}{5} = 0$

$x = \pm \frac{2}{\sqrt{5}}$

g: $16x^2 + \frac{16}{5} - \frac{80}{5} = 0$

$16x^2 - \frac{64}{5} = 0$

$x^2 = \frac{4}{5}$

$x = \pm \frac{2}{\sqrt{5}}$