① More on sheaves

i) The Hom sheaf

Let X be a top space, and $\mathcal{F}$ a sheaf on $\mathrm{Ouv}(X)$
Define the restriction for an open $U \subseteq X$
$$\mathcal{F}|_U$$
where for $V \subseteq U$ open,
$$\mathcal{F}|_U(V) := \mathcal{F}(V)$$

Now let $\mathcal{F}, \mathcal{G}$ be sheaves $\mathrm{Ouv}(X)^{op} \to \mathrm{Ab}$
Show that $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is an Abelian group of sheaves homo $\mathcal{F}|_U \to \mathcal{G}|_U$

So we have another presheaf $\mathcal{H}om : \mathrm{Ouv}(X)^{op} \to \mathrm{Ab}$

recall $\mathcal{F} \to \mathcal{G}$:
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$

Show that $\mathcal{H}om$ is actually a sheaf.

Pf. 1. $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is an Abelian group:
The morphism of sheaves becomes a compatible family of group homo, and group homo of Abelian groups form an Abelian group with pointwise operation.

2: $\mathcal{H}om$ is a sheaf:

Suppose $(s_i \in \mathrm{Hom}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i}))_{i \in I}$ satisfies where $U = \bigcup_{i \in I} U_i$
$$s_i|_{U_{ij}} = s_j|_{U_{ij}} \quad \forall i, j \in I$$
$s_i : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ sheaf homomorphism
$s_j : \mathcal{F}|_{U_j} \to \mathcal{G}|_{U_j}$
then $s : \mathcal{F}|_U \to \mathcal{G}|_U$ is the unique sheaf homo extension: just combine directly

② Spec R

Recall: Let Spec R be the set of all prime ideals of R.

Define $Z(I)$ to be the set of all prime ideals containing I; $Z(I) := \{ \mathfrak{p} : I \subseteq \mathfrak{p} \}$

Define a topology on Spec R by taking $Z(I)$ as the closed subsets

Denote the open sets by $D(r) := \{ \mathfrak{p} : r \notin \mathfrak{p} \}$ for $r \in R$ (open complement of $Z((r))$)

$\{ D(r) : r \in R \}$ is a basis for the Zariski top.

i) Describe Spec $\mathbb{Z}$, $Z(I)$ and $D(r)$

Ans: Prime ideals: $(0), (2), (3), (5), (7), \ldots$

$\therefore \operatorname{Spec} \mathbb{Z} = \{ (0) \} \cup \{ (p) : \text{prime } p \}$

$I = (n)$ since $\mathbb{Z}$ is PID.

$(n) \subseteq (p) \iff p \mid n$ prime decomposition

$\therefore Z(I) = Z((n)) = \{ (p) : p \mid n \}$

$n \notin (p) \iff p \nmid n$

$\therefore D(n) = \{ (p) : p \nmid n \}$

ii) Describe Spec $\mathbb{R}$, $Z(I)$ and $D(r)$

Ans: $\mathbb{R}$ is a field: (0) is the only prime ideal

$\therefore \operatorname{Spec} \mathbb{R} = \{ (0) \}$

For $I \neq (0)$, $Z(I) = \phi$, for $I = (0)$, $Z((0)) = \{ (0) \}$

For $r \neq 0$, $D(r) = \{ (0) \}$, for $r = 0$, $D(0) = \phi$

iii) Describe Spec $\mathbb{C}[x]$

Ans: Prime ideals: $(x-t)$, $\{0\}$ $\mathbb{C}[x]$ is PID

$\therefore \operatorname{Spec} \mathbb{C}[x] = \{ (0) \} \cup \{ (x-t) : t \in \mathbb{C} \}$

FTA $\Rightarrow$ $Z((f)) = \{ (x-t) : x-t \mid f \}$

$D(f) = \{ (x-t) : x-t \nmid f \}$

(3) Ringed spaces $(X, \mathcal{O}_X)$ and schemes

Recall : structure sheaf of a top space $X = \operatorname{Spec} R$

$\mathcal{O}_X(U) := \{ \varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U} \mid \varphi_{\mathfrak{p}} \in R_{\mathfrak{p}}$ & $\forall \mathfrak{p} \in U, \exists \mathfrak{p} \in V \subseteq U$ open, $g, h \in R$ s.t. $\forall \mathfrak{q} \in V, \varphi_{\mathfrak{q}} = \frac{g}{h} \in R_{\mathfrak{q}} \}$

E.g. i) X top space

$\mathcal{O}_X$ sheaf of continuous functions to $\mathbb{R}$

recall $C(S) = \{ f : S \to \mathbb{R} \text{ cont} \}$ forms a ring with pointwise addition & mult

$(f+g)(s) := f(s) + g(s)$

$(fg)(s) := f(s) \cdot g(s)$

For open $U \subseteq X$

$\mathcal{O}_X(U) := \{ f : U \to \mathbb{R} \text{ cont} \}$

E.g. ii) open $X \subseteq \mathbb{C}^n$ open complex subspace

$\mathcal{O}_X$ sheaf of holomorphic function to $\mathbb{C}$

For open $U \subseteq X$

$\mathcal{O}_X(U) := \{ h : U \to \mathbb{C} \text{ holomorphic} \}$

E.g. iii) Let $\mathbb{k}$ be alg closed field

$X \subseteq \mathbb{k}^n$ an affine variety

i.e. $X = V(I)$, I ideal in $\mathbb{k}[x_1, \dots, x_n]$

with the Zariski top

$\mathcal{O}_X$ sheaf of regular functions on X

i.e., open $U \subseteq X$,

$\mathcal{O}_X(U) := \{ U \to \mathbb{k} \text{ regular functions} \}$

Recall a regular function f can be stated as follows

$\forall x \in U$, $\exists x \in V \subseteq U$ open, $g_1, g_2 \in \mathbb{k}[x_1, \dots, x_n]$

s.t. $\forall t \in V$, $f(t) = \frac{g_1(t)}{g_2(t)}$ where $g_2(t) \neq 0$

iv) Let V be an affine scheme, $V = \operatorname{Spec} R$, R $\mathbb{k}$-alg

Show that $R \longrightarrow \mathbb{k}[x]/(x^2)$ corresponds to a tangent vector at a point

Pf. Let $\mathfrak{p} \in \operatorname{Spec} R$
Consider the local ring $R_\mathfrak{p}$ obtained via localisation, with the (unique) max ideal $m_\mathfrak{p}$.
Recall the tangent space of $\operatorname{Spec} R$ at $\mathfrak{p}$ is given by $(m_\mathfrak{p}/m_\mathfrak{p}^2)^* = \{\mathbb{k}\text{-lin } m_\mathfrak{p}/m_\mathfrak{p}^2 \to \mathbb{k}\}$
We want to show that there exists a bijection between
$(m_\mathfrak{p}/m_\mathfrak{p}^2)^*$ and $\operatorname{Hom}^{\mathfrak{p}}(\operatorname{Spec} \mathbb{k}[x]/(x^2), \operatorname{Spec} R)$
image is $\mathfrak{p}$

Note that if $f: \operatorname{Spec} R \to \operatorname{Spec} S$, $\mathfrak{p} \mapsto \mathfrak{q}$, then the corr ring homo $f^\#: S \to R$ has $\mathfrak{q}$ as the preimage of $\mathfrak{p}$.
In our case, $f^{\#-1}(x) = \mathfrak{p} \Rightarrow$ for $r \notin \mathfrak{p}$, $f^\#(r) \notin (x)$. So we can define $f^\#(r^{-1}) = f^\#(r)^{-1}$
$\operatorname{Spec} \mathbb{k}[x]/(x^2) \to \operatorname{Spec} R = \operatorname{Spec} \mathbb{k}[x]/(x^2) \to \operatorname{Spec} R_\mathfrak{p}$
$\Rightarrow$ WLOG, assume R is a local ring.
It suffices to show that there exists a bijection between
m/m^2 and $\operatorname{Local Hom}(R, \mathbb{k}[x]/(x^2))$ preimage of elements in max ideal is in max ideal

given a local homo $R \to \mathbb{k}[x]/(x^2)$, we obtain a $\mathbb{k}$-lin map
$m/m^2 \to (x)/(x^2) = \mathbb{k}$ no higher degree $\mathbb{k} \cdot x$
Given a $\mathbb{k}$-linear map $\varphi: m/m^2 \to \mathbb{k}$, construct a local homo h:
$h(a+b) = a + h(b)$ where $b \in m$

any $r \in R$
$= a+b$. $b \in m$

v) Let $\mathbb{k}$ be alg closed, R be an $\mathbb{k}$-algebra.
Suppose $\dim_\mathbb{k} R = 2$, i.e., R is the $\mathbb{k}$-vector space
$\mathbb{k}[x]/(x-a)(x-b)$

Classify the points in R.

Ans: When $a=b$, $R = \mathbb{k}[x]/(x^2)$
$\operatorname{Spec} R = \{(x)\}$ consists of a double point

When $a \neq b$, $R \cong \mathbb{k} \times \mathbb{k}$
$\operatorname{Spec} R \cong \operatorname{Spec}(\mathbb{k} \times \mathbb{k})$ consists of 2 points since $\mathbb{k}$ is a field.
$\mathbb{k} \times (0)$
$(0) \times \mathbb{k}$

vi) Let Spec R be an affine scheme over Spec k

i.e., there is a scheme morphism

$$\operatorname{Spec} R \to \operatorname{Spec} k$$

Show that R is indeed a k-algebra

$$\begin{array}{c} \qquad k \\ \qquad \downarrow \\ R \times R \to R \end{array}$$

Pf. We know that

$$\operatorname{Spec} R \to \operatorname{Spec} k \iff k \to R \quad \text{ring homs}$$

Since R is a comm ring, with $k \to R$, we define R as a k-vector space.
Hence R is an k-alg.