(1) Ringed spaces & locally ringed spaces

i) Let X be a top space
Recall the sheaf of cont funct on X
$\mathcal{O}_X(U) := \{ \text{cont } f : U \to \mathbb{R} \}$
What is the morphism of the ringed spaces
$(\varphi, \varphi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$?

Ans: $\varphi : X \to Y$ cont
$\forall V \subseteq Y$,
$\varphi^\# : \mathcal{O}_Y(V) \to \varphi_* \mathcal{O}_X(V)$
$(g : V \to \mathbb{R}) \mapsto (g \circ \varphi|_{\varphi^{-1}(V)} : \varphi^{-1}(V) \to \mathbb{R})$

ii) $X \subseteq \mathbb{C}^n$ open complex subspace
Recall the sheaf of holomorphic function to $\mathbb{C}$
$\mathcal{O}_X(U) := \{ h : U \to \mathbb{C} \text{ holomorphic} \}$
What is the morphism of the ringed spaces
$(\varphi, \varphi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$?

Ans: $\varphi : X \to Y$ holomorphic
$\forall V \subseteq Y$,
$\varphi^\# : \mathcal{O}_Y(V) \to \varphi_* \mathcal{O}_X(V)$
$(g : V \to \mathbb{C}) \mapsto (g \circ \varphi|_{\varphi^{-1}(V)} \to \mathbb{C})$

iii) Let $\mathbb{k}$ be alg closed field
$X \subseteq \mathbb{k}^n$ an affine variety
i.e. $X = V(I)$, I ideal in $\mathbb{k}[x_1, \dots, x_n]$
with the Zariski top
$\mathcal{O}_X$ sheaf of regular functions on X
$\mathcal{O}_X(U) := \{ U \to \mathbb{k} \text{ regular function} \}$
What is the morphism of the ringed spaces
$(\varphi, \varphi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$?

Ans: $\varphi : X \to Y$ polynomial map or
then precomposing a regular funct on Y gives
a regular funct on X.

iv) Let M be a smooth manifold,
i.e. inf differentiable manifold

locally $\mathbb{R}^n \to M$ diff

Define $\mathcal{O}_X(U) := C^\infty(U)$

Show that for any $p \in M$,

$$\mathcal{O}_{X,p} = \mathcal{O}_X(U)_{m_p}$$

where m_p is the maximal ideal of functions vanishing at p

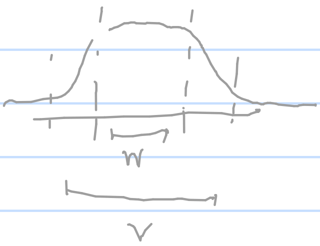
Pf. ($\supseteq$): Fractions of smooth functions are smooth.
$\mathcal{O}_X(U)_{m_p}$ consists of fractions of smooth functions defined on some nbh V of p.

($\subseteq$): $\forall$ germ $s_p \in \mathcal{O}_{X,p}$, $\exists$ open nbh $V \subseteq U$ s.t. s_p is well-defined.

Pick an open nbh $W \subseteq V$, and let $B(x)$ be the bump function s.t. $B_W(x) = 1$ $\forall x \in W$ and $B(x) = 0$ $\forall x \notin V$.

$s_p \cdot B$ is also a smooth function on U with the same germ.

extend to U
representative



v) Let R be a comm ring.
Show that the max ideals in R corr to closed points in $\operatorname{Spec} R$.

Pf. ($\Rightarrow$): Let m be a max ideal.
Then $Z(m)$ is a singleton because no any other prime ideals could contain m.
So m is a closed point

($\Leftarrow$): Suppose m is a closed point.
i.e., $m = Z(I)$ for some ideal I
$\Rightarrow$ No other prime ideal contains m
$\Rightarrow$ m is a max ideal

vii) $(\mathrm{Gal}(\bar{k}/k))$

Denote by Max Spec R the maximal ideals in R.

Let k be a field, possibly non-algebraic closed.
Show that there is a map (set-theoretic)
$\mathrm{Max\,Spec}(\bar{k}[X]) \twoheadrightarrow \mathrm{Max\,Spec}(k[X])$ (surjection)

Show that this maps restricts to a bijection
$\mathrm{Max\,Spec}(\bar{k}[X]) / \mathrm{Gal}(\bar{k}/k) \cong \mathrm{Max\,Spec}(k[X])$

Pf. Given a max. ideal M in $\bar{k}[X]$, we obtain an ideal $m := M \cap k$, which is also maximal.

Now let m be a max ideal in $k[X]$.
Then $m\bar{k}[X]$ is an ideal in $\bar{k}[X]$, which will then be contained in some max ideal M in $\bar{k}[X]$, and thus M is mapped to m under the map above.

Restriction to $\cong$:

Let M be an ideal in $\bar{k}[X]$ which restricts to m in $k[X]$.
finite M_i ∵ finite ext by Noether's Normalisation
Let M_i be all Galois conjugates of M
Then $\prod M_i$ is invariant under action of $\mathrm{Gal}(\bar{k}/k)$, hence is contained $m\bar{k}[X]$
$\therefore$ M_i's are the only maximal ideals containing m.

② Fibred product of schemes

Let S be a scheme.

Recall that a scheme over S is a scheme X with a mor of schemes $X \to S$.

A scheme over a field k is a scheme over $\operatorname{Spec} k$.

Def. X, Y schemes over S, $f : X \to S$, $g : Y \to S$.
A fibred product of X and Y over S is a scheme $X \times_S Y$ with mor $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ s.t.

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

is a pullback diag

Properties (from CT)
i) if $X \times_S Y$ exists, it is unique up to iso
ii) $X \times_S Y \cong Y \times_S X$
iii) $(X \times_S Y) \times_T Z \cong X \times_S (Y \times_T Z)$

i) Show that fibred products exist in AffSch, the cat of affine schemes, i.e. $\operatorname{Spec} A \times_{\operatorname{Spec}} \operatorname{Spec} B$ exists.

Pf. Claim: $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_C B)$

From lecture, we have $\mathrm{Sch}(X, \operatorname{Spec} A) \cong \mathrm{CRng}(A, \mathcal{O}_X(X))$ nat iso.
Since $A \otimes_C B$ is the pushout in CRng
i.e.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ A \arrow[r] \arrow[drr, bend right] & A \otimes_C B \arrow[dr, dashed, "\exists!"] \arrow[ul, phantom, "\ulcorner", very near start] & \\ & & \mathcal{O}_Z(Z) \end{tikzcd}$$

So from the natural iso, we obtain

$$\begin{tikzcd} Z \arrow[r, dashed, "\exists!"'] \arrow[rrr, bend left] \arrow[drr, bend right] & & \operatorname{Spec}(A \otimes_C B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ & & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} C \end{tikzcd}$$

ii) Let $C = k$ a field, $A = B = k[x]$

What is $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$?

Ans : $A \otimes_k B \cong k[x_1, x_2]$

$\therefore \operatorname{Spec}(A \otimes_k B) \cong \mathbb{A}^2$