① Algebra tools

Recall that the saturation of a homogeneous ideal I is
↳ gen by homo poly

$$\bar{I} = \{ f \in \mathbb{k}[x_0, \dots, x_n] \mid \exists k \geq 0 : (m_0)^k f \subseteq I \}$$

where m_0 is the ideal of polynomials without const, i.e. gen by $x_0, \dots, x_n$

which contains I.

i) Let $I \subseteq \mathbb{k}[x_0, \dots, x_n]$ be a homogeneous ideal.
Show that $I^{(d)} = \bar{I}^{(d)}$ for sufficiently large d

Pf. $\bar{I} = (f_1, \dots, f_r)$ since $\bar{I}$ is fin gen.
Then $(m_0)^{k_i} f_i \subseteq I$ by def of $\bar{I}$
For a high degree $f \in \bar{I}$, $f = a_1 f_1 + \dots + a_r f_r$
$\Rightarrow a_i$ has degree higher than k_i
$a_i f_i \in (m_0)^{k_i} f_i \subseteq I$
$\Rightarrow f \in I$

ii) (Lemma 21.2)
For homo ideals $I, J \subseteq \mathbb{k}[x_0, \dots, x_n]$, TFAE:
1. $\bar{I} = \bar{J}$
2. $I^{(d)} = J^{(d)}$ for sufficiently large d

Pf. $(1) \Rightarrow (2)$: $\bar{I} = \bar{J}$ so $\bar{I}^{(d)} = \bar{J}^{(d)}$
$\Rightarrow I^{(d)} = J^{(d)}$

$(2) \Rightarrow (1)$: For $f \in \bar{I}$, $\exists k \geq 0$, $(m_0)^k f \subseteq I$
$(m_0)^k f \cdot x_0^d \in I^{(d)} = J^{(d)}$
Since $x_0^d \in (m_0)^k$, so $(m_0)^{k+d} f \subseteq J$
$\Rightarrow f \in \bar{J}$

Recall an ideal $Q \subseteq R$ is primary $\Leftrightarrow$ $Q \neq R$ and $\forall\, x, y \in R$, $xy \in Q$, then $x \in Q$ $\vee$ $y^n \in Q$ for some $n > 0$.

Note that Q is primary $\Leftrightarrow$ $R/Q \neq 0$ and all zero divisors in R/Q are nilpotent.

Def. $I = \bigcap_{i=1}^{n} Q_i$ is called a minimal primary decomp where Q_i's are primary.
A minimal primary decomp is that $\text{rad}(Q_i)$ are distinct and $Q_i \not\supseteq \bigcap_{i \neq j} Q_j$ for each i

Def. $(I : u) = \{ r \in R \mid ru \in I \}$ is an ideal

Fact: For Noetherian ring, any I has a min primary decomp $I = \bigcap_{i=1}^{n} Q_i$
Define $P_i = \text{rad}(Q_i)$, called associated prime of I.
The set of all ass primes is denoted by $\text{Ass}(I)$

Algorithm: $P_i = \text{rad}(I : x_i)$ for $x_i \in R$.

iii) $k[x,y]$, $I = (x^2, xy)$

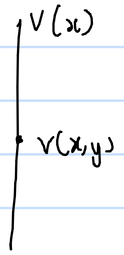
Find $\text{Ass}(I)$ and interpret the answer.

should compute for $u \notin I$

Ans:
$$(I : x) = (x, y)$$
$$\Rightarrow \text{rad}(I : x) = (x, y)$$
$$(I : y) = (x)$$
$$\Rightarrow \text{rad}(x) = (x)$$
$$(I : y^n) = (x)$$
$$\therefore \text{Ass}(I) = \{ (x), (x,y) \}$$

$V(I^n) = V(I)$

Interpretation: Note that $I = (x) \cap (x,y)^2$
so $V(I) = V(x) \cup V((x,y))$
$= V(x)$



Def. An ideal I is irreducible $\Leftrightarrow$ $I = J_1 \cap J_2 \Rightarrow I = J_1 \vee I = J_2$.

iv) Prove that every ideal in a Noetherian ring is a fin intersection of irr ideals.
(By contradiction, like we did for $V = \bigcup V_i$)

Pf. Suppose to the contrary, let S be the set of ideals that have no such decomposition.
Let $I \in S$. $I = J_1 \cap J_2$, $I \subsetneq J_1 \wedge I \subsetneq J_2$ since I itself is reducible
$\Rightarrow J_1 \in S$
Repeating, contradicting with Noetherian property.

v) Prove that in a Noetherian ring R, every irr ideal is primary.

Pf. Let I be irr.
Suppose $xy \in I$.
If $x \in I$, then I is primary.
If $x \notin I$, it suffices to show $y^n \in I$.
Construct the chain of ideals

$$(I:y) \subseteq (I:y^2) \subseteq (I:y^3) \subseteq \cdots$$

Since R is Noetherian, this terminates, i.e. $(I:y^{n+1}) = (I:y^n)$

Claim: $I = (I+(x)) \cap (I+(y^n))$ for the n above
($\subseteq$): obvious
($\supseteq$): Let $a \in (I+(x)) \cap (I+(y^n))$
Then $a \in I+(x)$
$\Rightarrow ay \in I$ since $xy \in I$
and $a \in I+(y^n)$
$\Rightarrow a + by^n \in I$ for some $b \in R$
$\Rightarrow ay + by^{n+1}$ I
$\Rightarrow by^{n+1} \in I$
$\Rightarrow b \in (I:y^{n+1}) = (I:y^n)$
$\Rightarrow by^n \in I$
So $(a+by^n) - by^n = a \in I$

I is irr and $x \notin I$, so $I \neq I+(x)$, hence $I = I+(y^n) \Rightarrow y^n \in I$

vi) Show that every ideal I in a Noetherian ring R has a primary decomposition, $I = \bigcap_{i=1}^{n} Q_i$, Q_i primary

Pf. Above results.

② Hilbert's polynomials & Bezout's Thm

Recall a Hilbert funct for a homogeneous ideal $I \subseteq \mathbb{k}[x_0, \dots, x_n]$ is h_I s.t. $\forall\, b \in \mathbb{N}$

$$h_I(b) = \dim \mathbb{k}^{(b)}[x_0, \dots, x_n] / I^{(b)}$$

Def. $\deg I = (\dim V(I))! \cdot LC(h_I)$

Bezout's Thm (Next lecture) Let $I \subseteq \mathbb{k}[x_0, \dots, x_n]$ be homogeneous. Let f be a homogeneous poly non-vanishing on any irr comp of I. Then $\deg(I + (f)) = \deg I \cdot \deg f$

i) Show that $h_0(b) = \binom{b+n}{n}$

Ans: The basis for $\mathbb{k}^{(b)}[x_0, \dots, x_n]$ consists of monomials of deg b.

$$x_0^{b_0} \cdot x_1^{b_1} \cdot \dots \cdot x_n^{b_n}$$

s.t. $b_0 + \dots + b_n = b$

$\rightsquigarrow$ How many non-neg sol to $b_0 + \dots + b_n = b$?

By stars-and-bars construction,

the no. of sol $= \binom{b+n}{n}$:

b ☆s $\quad$ ☆ | ☆ || ☆ $\dots$ ☆

n | s

total no. of object ☆ & | : $b+n$

Choose n position for the n bars

$\therefore \binom{b+n}{n}$

$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \dots \subset \mathbb{P}^n$ max length

ii) Find $\deg 0$

Ans:

$$\begin{aligned}\deg 0 &= (\dim \mathbb{P}^n)! \cdot LC\binom{b+n}{n} \\ &= n! \cdot LC\left(\frac{(b+n)!}{b!\, n!}\right) \\ &= n! \cdot LC\left(\frac{(b+n)\cdots(b+1)}{n!}\right) \\ &= n! \cdot \frac{1}{n!} = 1\end{aligned}$$

iii) Show $\deg(f) = \deg f$:

Ans: By Bezout's Thm.

$$\deg(0 + (f)) = \deg 0 \cdot \deg f$$

iv) Recall the Veronese curve X:

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$
$$(s:t) \mapsto (s^3 : s^2t : st^2 : t^3)$$

Find $h_{I(X)}(b)$.

Prove that $I(X) \neq (f,g)$, i.e., not generated by 2 homo poly.

(Fact: Veronese curve X is not planar)

Ans: The embedding corresponds to

$$\mathbb{k}[x_0, x_1, x_2, x_3] \to \mathbb{k}[s,t]$$
$$x_0 \mapsto s^3$$
$$x_1 \mapsto s^2t$$
$$x_2 \mapsto st^2$$
$$x_3 \mapsto t^3$$

which is indeed an iso

$$\mathbb{k}[x_0, x_1, x_2, x_3]/I(X) \xrightarrow{\cong} \mathbb{k}[s,t]$$

$\therefore$ $h_{I(X)}(b) = h_0(3b)$

Since if there is a deg b poly in $\mathbb{k}[x_0, x_1, x_2, x_3]$ then it is mapped to a deg $3b$ poly in $\mathbb{k}[s,t]$, which can be seen as a poly in $\mathbb{k}^{(3b)}[s,t]$.

$\therefore$ $h_0(3b) = \binom{3b+1}{1}$ by (i)

$= 3b+1$

Now since X is a curve, $\dim X = 1$

So $\deg I(X) = 1 \cdot 3 = 3$

By Bezout's Thm, if $I(X) = (f,g)$ $\quad (f) + (g) = (f,g)$

$$\deg I(X) = \deg f \cdot \deg g$$

$\Rightarrow$ WLOG, f is linear, which defines a plane, but it is well-known that Veronese curve is not planar.