① Divisors

- divisor :

Let $C \subseteq \mathbb{P}^2$ be a proj curve, g be a non-zero homo poly. Define $(g)_{div} = a_1 \cdot P_1 + \cdots + a_r \cdot P_r$

where $a_i = \deg (I(C) + (g))_{P_i}$ is the degree of $I(C)+(g)$ for the P_i component.

where $P_1, .., P_r$ are the intersection points of $C \cap V(g)$.

$(g)_{div}$ is called the divisor on $C \cap V(g)$.

Define the divisor group $\operatorname{Div} C = \mathbb{Z}C$

free Abelian gp on the set C

Write D_P for the coeff of a divisor D at P

i.e., $D = \sum_{P \in C} D_P \cdot P$ where $P \in C$

- degree of divisor :

The degree of D is $\deg D = \sum_{P \in C} D_P$

- order $\leq$:

Let D, E be divisors.

$D \leq E \iff D_P \leq E_P$? for all point P

Rk. 1. $(g)_{div}$ depends only on the class $g \in k[x_0, x_1, x_2] / I(C)$

2. $(g)_{div} \in \operatorname{Div} C$

By Bezout's Thm, $\deg (g) = \deg C \cdot \deg g$

$\Rightarrow (g)_{div} \geq 0$ (why $\deg (g) \geq 0$)

And $g(P) = 0 \Leftrightarrow (g)_{div_P} > 0$ for a $P \in C$

- Local parameter (uniformiser - generator of m)

rational funcs defined at P

A local parameter is a function $t \in \mathcal{O}_P$ generating m_P / m_P^2.

Every non-zero regular funct $f \in \mathcal{O}_P$ can be expressed as

$f = t^r g$ for some $r \in \mathbb{N}_0$, $g \in \mathcal{O}_P \setminus m_P$

$\hookrightarrow$ not vanish at P

i) Let f be a non-zero rational function on C.
Show that f is regular at point P $\Leftrightarrow$ $(f)_{\operatorname{div} P} \geq 0$

$\hookrightarrow$ well defined

Pf. Let t be a local parameter at P.
Since f is rational, $f = \frac{g}{h} = \frac{g}{k} / \frac{h}{k}$ for some homogeneous k not vanishing at P.

$g' := \frac{g}{k}$ $\quad h' := \frac{h}{k}$

$\therefore f = \frac{g'}{h'}$ for $g', h' \in \mathcal{O}_P$

Now since t is a local para,

$g' = t^r g''$, $\quad h' = t^s h''$

where $g'', h'' \in \mathcal{O}_P \setminus m_P$.

$\therefore f = t^{r-s} \frac{g''}{h''}$

$f'' := \frac{g''}{h''}$ is non-zero and regular, so $(f'')_{\operatorname{div} P} = 0$
(by Rk. on divisors)

By Thm in lecture, $(f)_{\operatorname{div} P} = (t^{r-s} f'')_{\operatorname{div} P}$
$= (t^{r-s})_{\operatorname{div} P} + (f'')_{\operatorname{div} P}$
$= (r-s) \cdot (t)_{\operatorname{div} P}$

Since $(t)_{\operatorname{div} P} > 0$, so $(f)_{\operatorname{div} P} \geq 0 \Leftrightarrow r-s \geq 0$
$\Leftrightarrow f$ is regular at P

ii) Let g, h be homogeneous poly where $(h)_{\operatorname{div}} \leq (g)_{\operatorname{div}}$.
Then $h \mid g$ in the quot ring $\mathbb{k}[x_0, x_1, x_2] / I(C)$.

Pf. It suffices $g = hf$ for some f homo poly
Let $r = \deg g - \deg h$,
we want to find f s.t. $\deg f = r$.

Note that $\frac{g}{x_0^r h}$ is regular on $\mathbb{A}^2 \cong U_0$, since $(h)_{\operatorname{div}} \leq (g)_{\operatorname{div}}$ ($x_0 \neq 0$)

$\Rightarrow (g)_{\operatorname{div}} - (h)_{\operatorname{div}} \geq 0$

$\Rightarrow \frac{g}{h}$ is regular on $\mathbb{A}^2$, by (i)

$\Rightarrow$ on $\mathbb{A}^2$, $\frac{g}{h} = k$ for some polynomial $k \in \mathbb{k}[x_1, x_2]$

Now by homogenisation,

$k = x_0^s \tilde{k}$ for some $s \in \mathbb{N}_0$.

$\therefore f = x_0^s \tilde{k}$.

② Elliptic curves

Thm. Let C be a cubic smooth irr planar curve, i.e. an elliptic curve.

Then
$$C \to Cl^0 C \qquad — \; Cl^0 C = \mathrm{Div}^0 C / P\mathrm{Div}\, C$$
$$P \mapsto P - P_0$$

is a bijection

deg 0 element in Div, i.e. $\sum a_i = 0$ $\qquad f = \frac{g}{h}$, same deg

Pf. Injectivity:

It suffices to show that if $P - P_0$ is a principal divisor, then it is 0.

Choose a coordinate sys s.t. $P_0 \in V(x_0)$.

Write $(x_0)_{div} = P_0 + P_1 + P_2$ — 3 intersection, not necessarily distinct

Suppose $P - P_0$ is principal,

i.e. $P - P_0 = (f)_{div} = (g)_{div} - (h)_{div}$ where $f = \frac{g}{h}$

We have
$$(gx_0)_{div} = (g)_{div} + (x_0)_{div} \quad \text{by Thm}$$
$$= P - P_0 + (h)_{div} + P_0 + P_1 + P_2$$
$$= (h)_{div} + P_1 + P_2 + P$$

$$\Rightarrow (gx_0)_{div} \geq (h)_{div}$$

By (ii), $h \mid gx_0$.

So $f = \frac{gx_0}{hx_0} = \frac{l}{x_0}$ where $(l)_{div} = P_1 + P_2 + P$.

Since there is only 1 line passing through P_1, P_2, and both $(x_0)_{div}$ and $(l)_{div}$ contains $P_1 + P_2$,

so $(x_0)_{div} = (l)_{div}$

$$\Rightarrow P = P_0 \Rightarrow P - P_0 = 0$$

- inflexions:

A point P on C is an inflexion pt $\Leftrightarrow$ $\det d^2 f|_p = 0$ for $I(C) = (f)$

det of Hessian of f at P

A point P on C (possibly non-cubic) (tangent) is an ordinary inflex point $\Leftrightarrow \exists$ a line l passes through P with $(l)_{div\,P} = 3$

Fact. The intersection no. of C and $\det d^2 f$ at an inflex pt P is 1 $\Leftrightarrow$ P is an ordinary inflexion pt

Thm. Every elliptic curve C has 9 inflexion pts.

Pf. We show that the inflex pts are intersection pts of C and $V(\det d^2 f)$.

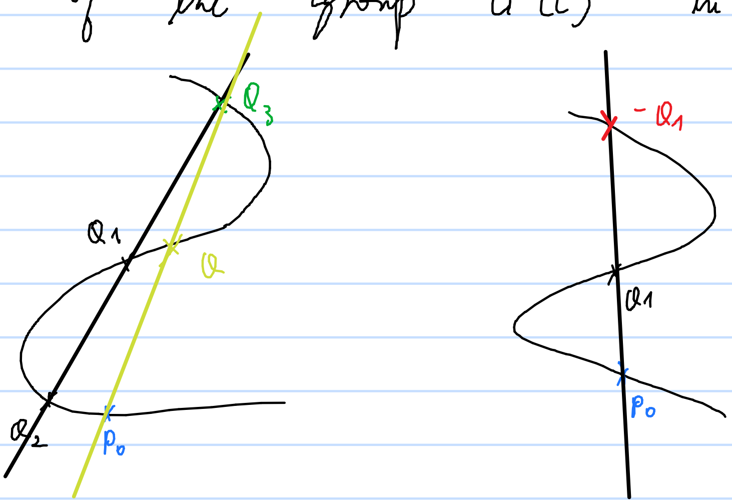
Note that $\det d^2 f$ is also cubic.

By Bézout's Thm, $\deg(I(C) + (\det d^2 f)) = 9 \Rightarrow (\det d^2 f)_{div} = P_1 + \dots + P_9$

The 9 inflex pts are distinct: since C is cubic, $(l)_{div\, P}$ is always 3, so all inflexion are ordinary.

Now by the fact above, P_i are distinct.

Visualisation of the group $Cl^\circ(C)$ in C:



1. Fix a pt P_0
2. $Q_1 + Q_2$: draw a line L through Q_1, Q_2

   L passes through Q_3

   draw a line L' through Q_3, P_0

   L' passes through Q

   $\rightsquigarrow Q_1 + Q_2 = Q$
3. $-Q_1$: draw a line L through Q_1, P_0

   L passes through $-Q_1$
4. identity : P_0