(4) Varieties & ideals

i) $f = x^2+y^2+1 \in \mathbb{R}[x,y]$

Show $I(V(f)) = (1) = \mathbb{R}(x,y)$

Pf. $x^2+y^2=-1$ has no sol $\Rightarrow V(f)=\phi$
And $I(\phi) = (1)$

ii) Every affine variety in $\mathbb{A}^2(\mathbb{R})$ is equal to $V(f)$ for some $f \in \mathbb{R}[x,y]$

Pf. Suppose $V(I)$, $I = (f_1, \ldots, f_n)$
Take $f = f_1^2 + \cdots + f_n^2$
since $f \in I$, obviously $V(I) \subseteq V(f)$
Now suppose $p \in V(f)$
i.e. $f_1^2(p) + \cdots + f_n^2(p) = 0$

$\mathbb{R}$ is integral domain

$\Rightarrow f_1^2(p) = 0, \ldots, f_n^2(p) = 0$
$\Rightarrow f_1(p) = 0, \ldots, f_n(p) = 0$
$\Rightarrow p \in V(I)$

iii) Show $\left\{\begin{matrix}\text{radical / prime / maximal} \\ \text{ideals of } R/I\end{matrix}\right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{\begin{matrix}\text{radical / prime / maximal} \\ \text{ideals of } R \text{ containing } I\end{matrix}\right\}$

Hint:
$\pi : R \to R/I$ natural quotient homo
1. $\forall$ ideal $J' \subseteq R/I$, $\pi^{-1}(J') = J \subseteq R$ is an ideal containing $I$
2. $\forall$ ideal $R \supseteq J \supseteq I$, $\pi(J) = J' \subseteq R/I$ is an ideal

Pf. 1. $J' = \{ j+I : j \in J\}$
$(J,+)$ is a subgroup of $(R,+)$ since $(J',+)$ is
And $(r+I)\cdot(j+I) = rj+I \in J' \Rightarrow rj \in J$
$\therefore J$ is an ideal
And $\forall i \in I$, $i+I = 0+I \in J' \Rightarrow i \in J \Rightarrow I \subseteq J$

2. $(J',+)$ is a subgroup of $(R/I,+)$ since $I \subset J \Rightarrow 0+J \in J'$.

This gives a bijection between ideals in $R/I$ and $R$ containing $I$

Radical:

$j^n \in J \Leftrightarrow j^n + I \in J' \Leftrightarrow (j+I)^n \in J'$

Prime:

$j_1 j_2 \in J \Leftrightarrow j_1 j_2 + I \in J' \Leftrightarrow (j_1+I)(j_2+I) \in J'$

Maximal:

Suppose $J' \subset R/I$ is maximal

$\forall$ ideal $K' \supset J'$, $K' = J'$ or $K' = R/I$

$\pi^{-1}(K') = K \supset J = \pi^{-1}(J')$ are ideals and $\pi^{-1}(R/I) = R$, $J$ is a maximal ideal in $R$

Similarly, $\pi(K) = K' \supset J' = \pi(J)$

iv) From (iii), show $R/J \cong (R/I)/J'$ : $J' = \pi(J)$

Pf. $\pi : R \to R/I$ induces

$\pi' : R/J \to (R/I)/J'$

Surjection: $\pi$ is quotient

Injection: Suppose $r+J \overset{\pi'}{\mapsto} 0$ in $(R/I)/J'$

i.e., $\pi'(r+J) = 0$ in $(R/I)/J'$

$\Rightarrow \pi(r) \in J'$

$\Rightarrow r \in J = \pi^{-1}(J')$

So $r+J = 0$ in $R/J$

v) The product $V \times W$ of affine varieties $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ is also an affine variety in $\mathbb{A}^{n+m}$.

Pf. $V = V(f_1, \ldots, f_r)$, $W = V(g_1, \ldots, g_s)$
where $f_1, \ldots, f_r \in \mathbb{k}[x_1, \ldots, x_n]$ and $g_1, \ldots, g_s \in \mathbb{k}[y_1, \ldots, y_m]$
so $f_1, \ldots, f_r, g_1, \ldots, g_s \in \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$

$$V \times W = V(f_1, \ldots, f_r, g_1, \ldots, g_s) \subseteq \mathbb{A}^{n+m}$$

vi) a) $V \subseteq \mathbb{A}^n$, $p \notin V$. Show that there is a poly $f \in \mathbb{k}[x_1, \ldots, x_n]$ s.t. $f(q) = 0$ $\forall q \in V$, but $f(p) = 1$.

b) $p_1, \ldots, p_r \notin V$. Show $\exists f_1, \ldots, f_r \in IV$, $f_i(p_j) = \begin{cases} 0 & i \neq j \\ 1 & j = i \end{cases}$

Pf. For (a):

$V \cup \{p\}$ is affine variety since it is attaching $x - p$ to the set of polynomials.
So $V \cup \{p\} \neq V \Rightarrow I(V \cup \{p\}) \neq I(V)$
otherwise $V(I(V \cup \{p\})) = V \cup \{p\}$
$= V(I(V)) = V$ — affine varieties

For (b):

$$I(V \cup \{p_1\} \cup \cdots \cup \{p_{r-1}\}) \neq I(V \cup \{p_1\} \cup \cdots \cup \{p_r\})$$

vii) (Nakayama's Lemma)

Let $A$ be a local ring (has unique maximal ideal) with a maximal ideal $\mathfrak{m}$; $N$ be a finitely generated $A$-mod s.t. $N\mathfrak{m} = N$. Then $N = 0$.

Pf. Let $x_1, \dots, x_n$ be generators of $N$.

Since $N\mathfrak{m} = N$, we have

$$x_j = x_1 a_{1j} + \cdots + x_n a_{nj}$$

for $a_{ij} \in \mathfrak{m}$

This is equivalent to

E is identity matrix

$$(x_1, \dots, x_n)(E - M) = 0 \quad \text{where } M = (a_{ij})$$

By multiplying the adjoint of $E - M$,

$$(x_1, \dots, x_n) \det(E - M) = 0$$

$$\Rightarrow x_j \det(E - M) = 0 \quad \forall j$$

Note that $\det(E - M) \in 1 + \mathfrak{m}$

$$\det\begin{pmatrix} 1-a_{11} & & \\ & 1-a_{22} & \\ & & \ddots \end{pmatrix} = 1 \pm a_{ij} \cdots$$

$$\Rightarrow x_j = 0 \quad \forall j$$

$-1 \notin \mathfrak{m}$

$\therefore N = 0$

② Nullstellensatz $\quad \Bbbk$ alg closed

i) (3) $I(V(J)) = \sqrt{J}$

(2) $V(J) = \phi \iff 1 \in J$

(1) max ideal $\leftrightarrow$ points

Show (3) $\Rightarrow$ (2) & (3) + (2) $\Rightarrow$ (1)
(Hint: Show $V(M)$ is a singleton)

Pf.

(3) $\Rightarrow$ (2):

Suppose $V(J) = \phi$

$\sqrt{J} = I(V(J)) = I(\phi) = \Bbbk[X]$

so $1 \in \sqrt{J} \Rightarrow 1 \in J$

$1 \in J \Rightarrow V(J) = \phi$ is trivial (done last time)

(2) + (3) $\Rightarrow$ (1):

A maximal ideal is a prime ideal which is a radical ideal.

so $I(V(M)) = M$

Claim: $V(M)$ is a singleton

Suppose $M \subsetneq I$, since $M$ is maximal, $I = R = \Bbbk[x_1, \dots, x_n]$
$V(I) = \phi$

By (2), $V(M) \neq \phi$

$V(M) \supset V(I)$
$\Rightarrow \phi \subset V(M)$

$V(M)$ can only have empty proper affine subvar.
Note that any singleton $\{p\}$ corresponds to $(x-p)$, a maximal ideal, so if $M$ has any subset, it is a variety.

$\therefore$ $M$ has only empty proper subset
$\Rightarrow$ singleton

$M \overset{V}{\mapsto} V(M)$ singleton (point)

$V(M) \overset{I}{\mapsto} (x-p) = M$ max ideal

singleton

ii) What if $\mathbb{k}$ not alg closed?

Counterexample: $f = x^2+1 \in \mathbb{R}[x]$

$\mathbb{R}[x]$ is UFD

(3) $V(f) = \phi$

$f$ is irr so if $g^n \in (f)$ $f \mid g^n \Rightarrow f \mid g$

$I(V(f)) = I(\phi) = \mathbb{R}[x]$

$\sqrt{(f)} = (f)$

(2) $1 \notin (f)$ even though $V(f) = \phi$

(1) $(f)$ is a maximal ideal since it is a prime ideal. But $(f)$ is not of the form $(x-p)$

iii) Coroll.

$V(I)$ is finite $\Leftrightarrow \mathbb{k}[x_1,\dots,x_n]/I$ is fin dim vector space over $\mathbb{k}$

Hint: write a point $p_i$ as in coordinate form $p_i = (a_{i1}, \dots, a_{in})$

In this case, $|V(I)| \leq \dim_{\mathbb{k}}(\mathbb{k}[x_1,\dots,x_n]/I)$

Pf. $(\Rightarrow)$: Suppose $V(I) = \{p_1,\dots,p_r\}$ is finite:
Write $p_i = (a_{i1}, \dots, a_{in})$ coordinate
and define $f_j := \prod_{i=1}^{r} (x_j - a_{ij})$, for $j = 1,\dots,n$

$\Rightarrow f_j \in I(V(I))$ because $f_j$ vanishes at $p_i$

$\Rightarrow f_j^k \in I$ for some $k \in \mathbb{N}_{>0}$ as $I(V(I)) = \sqrt{I}$

Let $\bar{f_i}$ denote the image of $f_i$ in $\mathbb{k}[x_1,\dots,x_n]/I$

we have $\bar{f_j}^k = 0 \Rightarrow (\overline{x_j - a_{ij}})^{rk} = 0$

e.g. $x^3 = x^2+x$
then $x^4 = x(x^3) = x(x^2+x)$
$= x^3+x^2$
$= x^2+x^2+x$

$\Rightarrow \bar{x_j}^{rk}$ is $\mathbb{k}$-lin com of $1, \bar{x_j}, \bar{x_j}^2 \dots \bar{x_j}^{rk-1}$

Inductively, $\forall s \in \mathbb{N}_{\geq 0}$, $\bar{x_j}^s$ is $\mathbb{k}$-lin com of

so $\{\bar{x_1}^{m_1}, \dots, \bar{x_n}^{m_n} : m_i < rN\}$ is a generating set for $\mathbb{k}[x_1,\dots,x_n]/I$, which is finite

($\Leftarrow$) : Suppose $\mathbb{k}[x_1, \dots, x_n]/I$ is fin dim.

Let $p_1, \dots, p_r \in V(I)$ may have more pts

By ① (vi), we can take $f_1, \dots, f_r \in \mathbb{k}[x_1, \dots, x_n]$

s.t. $f_i(p_j) = \begin{cases} 0 & , i \neq j \\ 1 & , i = j \end{cases}$

$\bar{f_i}$ image in $\mathbb{k}[x_1, \dots, x_n]/I$.

Suppose $\lambda_1 \bar{f}_1 + \dots + \lambda_r \bar{f}_r = 0$ in $\mathbb{k}[x_1, \dots, x_n]/I$, $\lambda_i \in \mathbb{k}$, thing in $I$ vanishes at $p_j$

then $\lambda_1 f_1 + \dots \lambda_r f_r \in I$

$\Rightarrow \lambda_1 f_1(p_j) + \dots + \lambda_r f_r(p_j) = 0$

$\Rightarrow \lambda_j = 0$

so $\bar{f_i}$ are lin indep

$\Rightarrow r \leq \dim_{\mathbb{k}} (\mathbb{k}[x_1, \dots, x_n]/I)$

$$-x^2+y^2 = (y-x)(y+x)$$
$$x^2-(iy)^2 = (x-iy)(x+iy)$$

IV) $I = (\underset{f}{-x^2+y^2},\ \underset{g}{x^2+y^2}) \in \mathbb{C}[x,y]$

a) Find $V(I)$ via resultant:

$$\mathrm{Res}(f,g;x) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ y^2 & 0 & y^2 & 0 \\ 0 & y^2 & 0 & y^2 \end{pmatrix}$$
$$= 4y^2$$

$(0,0)$ is the only solution

so $V(I) = \{(0,0)\}$

b) Calculate $\dim_{\mathbb{C}}\left(\mathbb{C}[x,y]/I\right)$:

$f+g = 2y^2 \Rightarrow y^2 = \bar{0}$ in $\mathbb{C}[x,y]/I$

$f-g = -2x^2 \Rightarrow x^2 = \bar{0}$ in $\mathbb{C}[x,y]/I$

So any term with multiples of $x^2$ or $y^2$ is reduced mod $I$.

e.g. $x^3 \in \mathbb{C}[x,y] \rightsquigarrow \bar{0}$ in $\mathbb{C}[x,y]/I$

$\vdots$

So $\{1, x, y, xy\}$ is a basis for the vector space $\mathbb{C}[x,y]/I$.

$$\therefore \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = 4$$

## 3) Irreducibility

**Def.** An affine variety $V \subseteq \mathbb{A}^n$ is reducible $\Leftrightarrow$ $V = V_1 \cup V_2$ where $V_i$ are affine varieties in $\mathbb{A}^n$ and $V_i \neq V$.

Otherwise, $V$ is irreducible.

i) Prove that an affine variety $V$ is irreducible $\Leftrightarrow$ $I(V)$ is a prime ideal.

(Hint: use contraposition, so easier to make use of def of irreducible variety)

Pf. $(\Rightarrow)$:

Suppose $I(V)$ is not a prime ideal. Then $\exists\ fg \in I(V)$, with $f \notin I(V)$ and $g \notin I(V)$.

Then
$$\begin{aligned} V &= V(I(V)) && \text{since } V \text{ is variety. Galois corr} \\ &\subseteq V(f \cdot g) \\ &= V(f) \cup V(g) && \text{proved in last tutorial} \end{aligned}$$

So $V = (V \cap V(f)) \cup (V \cap V(g))$

where $V \cap V(f) \neq V$ and $V \cap V(g) \neq V$

$\therefore V$ is reducible

$(\Leftarrow)$: Suppose $V = V_1 \cup V_2$, $V_i \neq V$

Then $V_i \subsetneq V \Rightarrow I(V_i) \supsetneq I(V)$

$\Rightarrow \exists\ f_i \in I(V_i)$, $f_i \notin I(V)$

Then $f_1 f_2 \in I(V_1) \cap I(V_2) = I(V_1 \cup V_2) = I(V)$

$\Rightarrow I(V)$ is not a prime ideal