Last time:

$V(I)$ is finite $\iff$ $\Bbbk[x_1,\ldots,x_n]/I$ is fin dim vector space over $\Bbbk$

In this case, $|V(I)| \leq \dim_{\Bbbk}(\Bbbk[x_1,\ldots,x_n]/I)$

E.g.

iv) $I = (\overset{f}{\overset{\shortparallel}{-x^2+y^2}}, \overset{g}{\overset{\shortparallel}{x^2+y^2}}) \in \mathbb{C}[x,y]$

$-x^2+y^2 = (y-x)(y+x)$

$x^2-(iy)^2 = (x-iy)(x+iy)$

a) Find $V(I)$ via resultant:

$$\mathrm{Res}(f,g;x) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ y^2 & 0 & y^2 & 0 \\ 0 & y^2 & 0 & y^2 \end{pmatrix} = 4y^2$$

$(0,0)$ is the only solution

so $V(I) = \{(0,0)\}$

b) Calculate $\dim_{\mathbb{C}}(\mathbb{C}[x,y]/I)$:

$f+g = 2y^2 \Rightarrow y^2 = \bar{0}$ in $\mathbb{C}[x,y]/I$

$f-g = -2x^2 \Rightarrow x^2 = \bar{0}$ in $\mathbb{C}[x,y]/I$

So any term with multiples of x^2 or y^2 is reduced mod I.

e.g. $x^3 \in \mathbb{C}[x,y] \rightsquigarrow \bar{0}$ in $\mathbb{C}[x,y]/I$

$\vdots$

So $\{1, x, y, xy\}$ is a basis for the vector space $\mathbb{C}[x,y]/I$.

$\therefore \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = 4$

① Irreducibility

Recall :

Def. A non-empty top space V is reducible $\Leftrightarrow$

$V = V_1 \cup V_2$ where $V_i \subsetneq V$ are two closed proper subsets. Otherwise, irreducible.

i) Show that the following are equiv :

1) V is irreducible

2) any 2 non-empty open subsets of V has non-empty intersection

3) Every non-empty open subset of V is dense

Pf. (1) $\Leftrightarrow$ (2) : Suppose U_1, U_2 are 2 non-empty open subsets
$V \setminus U_1$, $V \setminus U_2$ are non-empty closed proper

Then $U_1 \cap U_2 = \phi \Leftrightarrow V = V \setminus U_1 \cup V \setminus U_2$

(2) $\Leftrightarrow$ (3) : Let U be non-empty open.

$\bar{U} \neq V \Leftrightarrow V = \bar{U} \cup V \setminus U$

ii) Classify Irreducible Haus spaces.

Pf. Let X be an irreducible Haus space.

Haus $\Leftrightarrow$ for any $x \neq y \in X$, $\exists$ $U_x \ni x$ and $U_y \ni y$, $U_x \cap U_y = \phi$.

Yet X is irredu $\Leftrightarrow$ $\forall$ $U_1, U_2 \in X$, $U_1 \cap U_2 \neq \phi$.

So the only case both can happen is $X = \{*\}$.

Rk. This says Zariski top is often not Haus

iii) Let F and G be poly in $k[x,y]$ with no common factor. Then $V(F,G) = V(F) \cap V(G)$ is a fin set of points.

Pf.

No common factor $\Rightarrow$ no common component.

It suffices to count the no. of common root.

Note that $p \in V(F) \cap V(G)$

Then we proceed by considering $\mathrm{Res}(F,G;y)$, which is a polynomial, has fin root y_0. For each y_0, we only have fin no. of x_0 s.t. (x_0, y_0) is a common root.

The resultant poly has only finitely many sol

iv) a) Let $F \in \mathbb{k}[x,y]$ be an irr poly s.t. $V(F)$ is infinite.

Show: $I(V(F)) = (F)$

$V(F)$ is an irr affine var of $\mathbb{A}^2$

b) Show $F = y^2 + x^2(x-1)^2 \in \mathbb{R}[x,y]$ is irr

but $V(F)$ is reducible. (Hint: suppose $F = GH$)

Pf. (a): Let $G \in I(V(F))$

Then $V(G) \supseteq V(I(V(F))) = V(F)$

$V(F)$ is infinite $V(F,G) = V(F) \cup V(G)$ is infinite

By (iii), $F \mid G$ since F is irr, so $(F) \supseteq I(V(F))$

Now $(F) \subseteq I(V(F))$ Galois corr

so $(F) = I(V(F))$

Now (F) is prime ideal since F is irr and $\mathbb{k}[x,y]$ is UFD,

so $V(F)$ is irr recall V irr $\iff I(V)$ prime

(b): Suppose to the contrary F is reducible.

$$F = y^2 + x^2(x-1)^2 = GH, \quad G, H \in \mathbb{R}[x,y]$$

Case I: WLOG, G contains a y^2 term while H contains only x^n terms,

Then GH contains $x^n y^2$ terms $\lightning$

Case II: Both G and H has a y term.

i.e. $G = G_1 y + G_2$, $H = H_1 y + H_2$

Then GH

$= G_1H_1 y^2 + (H_2 G_1 + G_2 H_1) y + H_2 G_2$

$\Rightarrow G_1 H_1 = 1$ so they are constants

And $H_2 G_1 + G_2 H_1 = 0$ where $G_1^{-1} = H_1 \in \mathbb{R}$ now

$\Rightarrow H_2 G_2 = \dfrac{-G_2^2}{G_1^2} = x^2(x-1)^2$, $\lightning$ $-G_2^2 < 0$

$\therefore F$ is irr

Yet $V(F) = \{(0,0), (1,0)\} = \{(0,0)\} \cup \{(1,0)\}$

$= V(x^2+y^2) \cup V((x-1)^2+y^2)$

Rk. $V(F)$ is fin

Recall: Noetherian space X

$$X_1 \supseteq X_2 \supseteq \cdots \qquad \text{closed}$$

$$\exists\ m \text{ s.t. } X_m = X_{m+1}$$

v) Show a Noetherian space X is quasi-cpt
i.e. every open cover of X has a fin subcover

Pf. Suppose X is not quasi-cpt.
Pick U_0 in the open cover
Since U_0 cannot cover whole X, $\exists\ x \notin U_0$.
Pick $U_x \ni x$
$U_1 := U_0 \cup U_x$
$\vdots$

Then $X \setminus U_0 \supseteq X \setminus U_1 \supseteq \cdots$
is a descending chain of closed set in X

vi) Show for any Noetherian space X,
$X = X_1 \cup \cdots \cup X_n$
where X_i are irr spaces.

decompose into irr components

Pf. Existence:
Let $S = \{\text{affine var } V \subseteq \mathbb{A}^n : V \text{ has no such decomp}\}$.
It suffices to show $S = \emptyset$.
Suppose not, i.e., $S \neq \emptyset$.
By the previous remark, S has a minimal member, denote by V_0.
Write $V_0 = V_1 \cup V_2$, $V_i \subsetneq V_0$
By minimality of V_0, $V_i \notin S$, so $V_i = \bigcup_j V_{ij}$
with V_{ij} being irreducible.
Then $V_0 = \bigcup_{i,j} V_{ij}$, contradiction

Uniqueness:
Suppose $V_1 \cup \cdots \cup V_m = V = W_1 \cup \cdots \cup W_s$ are two such decomposition.
Then $V_i = V_i \cap V = \bigcup_{j=1}^{s} V_i \cap W_j$. Since V_i is irreducible, so $V_i = V_i \cap W_j$ for some j, this means $V_i \subseteq W_j$. Similarly, $W_j \subseteq V_{i'}$ for some i'. Then $V_i \subseteq V_{i'}$, so $V_i = V_{i'}$.
Hence $m = s$.

③ Poly function & Poly map

i) $F := y^2 - x^2(x+1)$, $W := V(F)$

Let $\varphi : \mathbb{A}^1 \to W$
$$t \mapsto (t^2-1,\ t(t^2-1))$$

Show φ is bijective except that $\varphi(\pm 1) = (0,0)$

Pf. Injective:
$$\varphi(t) = \varphi(s)$$
$t^2-1 = s^2-1$ and $t^3 - t = s^3 - s$

$(t-s)(t+s) = 0$ $\quad$ $(t-s)(t^2+ts+s^2-1) = 0$

$\Rightarrow t = s$ or $t = -s$ $\quad$ $\Rightarrow t = s$ or $t^2+ts+s^2 = 1$

So for $t = -s$ and $t^2+ts+s^2 = 1$ hold, $t = 1$, $s = -1$ (vice versa)

So $\varphi(1) = \varphi(-1) = (0,0)$, otherwise $t = s$, φ is injective.

Surjective:
$$(b_1, b_2) \in W$$
$$\Rightarrow b_2^2 = b_1^2(b_1+1)$$
$\Rightarrow b_1$ is determined by b_2

Since $t \mapsto t(t^2-1)$ is surjective, done.

ii) a) Show $\left\{ \begin{array}{l} \text{affine var /} \\ \text{irr subvar /} \\ \text{pts} \quad \text{of } V \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{radical /} \\ \text{prime /} \\ \text{maximal ideals of } \mathbb{k}[V] \end{array} \right\}$

b) Let $V \subseteq \mathbb{A}^n$, $W \subseteq V$ subvariety

Show every poly funct on V restricts to a poly funct on W.

Pf. (a): Recall: a coor ring $\mathbb{k}[V] \cong \mathbb{k}[x_1, \dots, x_n]/I(V)$

Suppose J is an ideal containing $I(V)$.
Then $V(J) \subset V(I(V)) = V \Rightarrow V(J)$ subvar of V

Apply Nullstellensatz on varieties

i.e. $I^X(V^X(J)) = \sqrt{J}$

(b) : $f : V \to \mathbb{k}$ poly funct

$\Rightarrow \exists F \in \mathbb{k}[x_1, .., x_n]$

$f(p_1, .., p_n) = F(p_1, .., p_n)$

$\forall (p_1, .., p_n) \in V$

Since $W \subset V$, $f(q_1, ..., q_n) = F(q_1, .., q_n)$

$\forall (q_1, ..., q_n) \in W$.