(1) Product variety

Recall: $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$

we have $V \times W \subseteq \mathbb{A}^{n+m}$

i) Does $V \times W$ satisfy the uni prop of product ?

Ans: Take polynomial map as morphism : yes.

$$
\begin{tikzcd}
 & Z \arrow[d, dashed, "f \ \exists!"] \arrow[ddl, bend right, "f_V"'] \arrow[ddr, bend left, "f_W"] & \\
 & V \times W \arrow[dl, "\pi_V"'] \arrow[dr, "\pi_W"] & \\
x_1, \dots, x_n \quad V & & W \quad y_1, \dots y_m
\end{tikzcd}
$$

$f = (f_1, \dots f_n)$

poly format

$\mathbb{k}[Z]$ coor ring of Z.

In previous lecture, we know

poly map $f_W : Z \to W \iff f_W^* : \mathbb{k}[W] \to \mathbb{k}[Z]$

$f_V : Z \to V \iff f_V^* : \mathbb{k}[V] \to \mathbb{k}[Z]$

as in the proof $f_W^*(y_j) = f_{Wj}$

i.e., f_W^* is determined by image of coor under f_W^*

Now f_{Wj} , f_{Vj} determine

$\mathbb{k}[x_1, \dots, x_n, y_1, \dots, y_m] / (I(V \times W)) \longrightarrow \mathbb{k}[Z]$

by concatenation. $\cong \mathbb{k}[x_1 \dots x_n, y_1 \dots, y_m]$

So we have

$f : Z \longrightarrow V \times W$ a poly map

Note that $\pi_V f = f_V$, $\pi_W f = f_W$

so $f(p) = (f_V(p), f_W(p)) \quad \forall p \in Z$

ii) Describe the image of the poly map

$$f : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$$
$$t \mapsto (\underset{f_1}{t^2-1}, \underset{f_2}{t^3-1})$$ by a variety

Ans: By def, $\Gamma_f = \{(t,b) : b = f(t)\}$

$\rightsquigarrow (x, y) = (f_1(t), f_2(t))$

$(0,0) = (f_1(t) - x, f_2(t) - y)$

So

$$\Gamma_f = V(t^2 - 1 - x, t^3 - 1 - y)$$

To find common sol, calculate — fix x, y as 'const'

$$\text{Res}(t^2-1-x, t^3-1-y; t)$$
$$= \det\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1-x & 0 & 1 & -1 & 0 \\ 0 & -1-x & 0 & -y & -1 \\ 0 & 0 & -1-x & 0 & -y \end{pmatrix}$$
$$= y^2 - x^2 - x^3$$

So finding the common sol is equiv to setting $y^2 - x^2 - x^3 = 0$

$\therefore \text{im} = V(y^2 - x^2 - x^3)$

iii) Describe the closure of the image of using gröbner basis

$$f : \mathbb{A}^2 \to \mathbb{A}^3$$
$$(s,t) \mapsto (s^2-t^2, 2st, s^2+t^2)$$

Idea: $\operatorname{im} f = \pi_2(\Gamma_f)$

Let $\Gamma_f = V(J)$.

By Thm 10.5 in the lectures, $V \subseteq \mathbb{A}^{n+m}$, $I(\pi(V)) = I(V) \cap \mathbb{k}[y_1, \dots, y_m]$

$\Rightarrow I(\pi(V(J))) = \sqrt{J} \cap \mathbb{k}[y_1, \dots, y_m]$

$\overline{\pi(V(J))}$

$= V(I(\pi(V(J))))$ recall VI is closure operator

$= V(\sqrt{J} \cap \mathbb{k}[y_1, \dots, y_m])$ e.g. $x^3 = x^2 \cdot x^2$

Now suppose $S \subseteq \mathbb{k}[x_1, \dots, x_n, y_1, \dots, y_m]$

gröb basis for S is certain generating set of S.

Consider $S \cap \mathbb{k}[y_1, \dots, y_m]$

By computing Gröbner basis for S with lexicographical order $x \succ y$

Then we obtain Gröbner basis $S \cap \mathbb{k}[y_1, \dots, y_m]$

$G' = G \cap \mathbb{k}[y_1, \dots, y_m]$

term has no x stays no x

Ans: $\Gamma_f = V(\underbrace{s^2-t^2-x}_{h_1}, \underbrace{2st-y}_{h_2}, \underbrace{s^2+t^2-z}_{h_3})$

gröbner

$g = h_1k_1 + h_2k_2$

$k_1 := \frac{1}{2}(h_1+h_3) = s^2 - \frac{1}{2}x - \frac{1}{2}z$

$k_2 := \frac{1}{2}h_2 = st - \frac{1}{2}y$

$k_3 := \frac{1}{2}(h_1 - h_3) = t^2 + \frac{1}{2}x - \frac{1}{2}z$

$S(k_1, k_2) = t(s^2 - \frac{1}{2}x - \frac{1}{2}z) - s(st - \frac{1}{2}y) = -\frac{1}{2}tx - \frac{1}{2}tz + \frac{1}{2}sy$

$S(k_2, k_3) = t(st - \frac{1}{2}y) - s(t^2 + \frac{1}{2}x - \frac{1}{2}z) = -\frac{1}{2}ty - \frac{1}{2}sx + \frac{1}{2}sz$

$\rightsquigarrow$ so we add $\underbrace{sy - tx - tz}_{k_4}$, $\underbrace{sx - sz + ty}_{k_5}$ into our collection

Now, $S(k_1, k_4) = y\,k_1 - s\,k_4 = -\frac{1}{2}xy - \frac{1}{2}yz + stx + stz$

$= (x+z)(-y + 2st) = 2(x+z)k_2 \to 0$

$S(k_1, k_5) = x\,k_1 - s\,k_5 = -\frac{1}{2}x^2 - \frac{1}{2}xz + s^2z - sty$

$= -\frac{1}{2}x^2 + z\,k_1 + \frac{1}{2}z^2 - sty$

$\to -\frac{1}{2}x^2 + z k_1 + \frac{1}{2}z^2 - y k_2 - \frac{1}{2}y^2 \to -\frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{2}z^2$

$S(k_2, k_4) = y\,k_2 - t\,k_4$

$= -\frac{1}{2}y^2 + t^2x + t^2z = -\frac{1}{2}y^2 - t(-tx - tz + sy) + syt$

$= t k_2 - t k_4 \to 0$

$S(k_2, k_5) = x k_2 - t k_5 = -\frac{1}{2}xy - stz - t^2y = -y(t^2 + \frac{1}{2}x - \frac{1}{2}z) + \frac{1}{2}yz - stz$

$= -y k_3 - z k_2 \to 0$

$szy - tzx - tz^2$

$$S(k_4, k_5) = x k_4 - y k_5$$
$$= -tx^2 - txz + syz - ty^2$$
$$= z k_4 + tz^2 - t^2y^2 - tx^2$$
$$= -tx^2 - ty^2 + tz^2$$

So we add $\underset{k_6}{\underset{\parallel}{x^2}} + y^2 - z^2$ into our collection

One checks that $G = \{k_1, \dots, k_6\}$ is a Gröbner basis

$$G \cap k[y_1, y_2, y_3] = k_6$$

$$\therefore \ \operatorname{im} f = V(x^2 + y^2 - z^2)$$

Rk. Gröbner basis

$\{g_1, \dots, g_m\}$ is Gröbner basis of the ideal

$I = (g_1, \dots, g_m) \Leftrightarrow \forall \ 0 \neq f \in I,$

$\exists \ g_i$ with $LT(g_i) \mid LT(f)$

$$S(f,g) = \frac{LCM}{LT(f)} f - \frac{LCM}{LT(g)} g$$

② Projective variety

Recall $\mathbb{P}(V) = (V \setminus \{0\}) / \sim$, scalar mult

where $u \sim v \iff \exists\, k \in \mathbb{k} \setminus \{0\},\ v = ku$

In particular, $\mathbb{P}^n := \mathbb{P}(\mathbb{k}^{n+1})$

$= \{$ lines thro $(0, \dots, 0)$ in $\mathbb{k}^{n+1}\}$

$\simeq \{ (x_0 : \cdots : x_n),\ x_i \in \mathbb{k}$ not all $0 \}$

$U_i := \{ (x_0 : \cdots : x_n) \in \mathbb{P}^n : x_i \neq 0 \}$

$H_i := \{ (x_0 : \cdots : x_n) \in \mathbb{P}^n : x_i = 0 \}$ — hyperplane at ∞

Note that $H_i \cong \mathbb{P}^{n-1}$

$(x_0, \dots, x_{n-1}, 0) \mapsto (x_0, \dots, x_{n-1})$

and $U_i \cong \mathbb{A}^n$

$(x_0 : \cdots : x_n) \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right)$

and $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$

and $\mathbb{P}^n = U_n \cup H_n$

A proj var is defined as
$V(f) = \{ (x_0 : \cdots : x_n) \in \mathbb{P}^n : f(x_0, \dots, x_n) = 0 \}$

i) What is a) $\mathbb{P}^0$
b) $\mathbb{P}^1$
c) $\mathbb{P}^2$?

Ans: a) $\mathbb{P}^0 = \mathbb{P}(\mathbb{k}^0 = \{*\}) = \{*\}$

b) $\mathbb{P}^1 = U_1 \cup H_1 = \{ (x_0 : x_1) : x_1 \neq 0 \} \cup \{ (x_0 : 0) \}$
$= \{ (x : 1) : x \in \mathbb{k} \} \cup \{ (1 : 0) \}$

c) $\mathbb{P}^2 = U_2 \cup H_2 = \{ (x_0 : x_1 : x_2) : x_2 \neq 0 \} \cup \{ (x_0 : x_1 : 0) \}$
$= \{ (x : y : 1) : x, y \in \mathbb{k} \} \cup \{ (x : y : 0) : (x : y) \in \mathbb{P}^1 \}$