① Projective Variety

More examples :

i) a) line $L : y = mx + c$ in $\mathbb{A}^2$

Identify $\mathbb{A}^2 \cong U_3 \subseteq \mathbb{P}^2$

What does L corresponds to in $\mathbb{P}^2$?

Ans : Recall : $\mathbb{A}^n \cong U_i \subseteq \mathbb{P}^n$

$$(x_1, \cdots, x_n) \mapsto (x_1 : \cdots : 1 : \cdots : x_n)$$
$$\left(\frac{x_1}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right) \leftarrow (x_0 : x_1 : \cdots : x_n)$$

So $(x : y : z) \mapsto \left(\frac{x}{z}, \frac{y}{z}\right)$

So L corresponds to $(x : y : z)$ satisfying $\frac{y}{z} = m\frac{x}{z} + c \Leftrightarrow y = mx + cz$, $z \neq 0$

b) What is $S = \{(x : y : z) \subseteq \mathbb{P}^2 \mid y = mx + cz\} \cap H_3$?
Interpret S.

Ans : $H_3 = \{(x : y : z) \subseteq \mathbb{P}^2 : z = 0\}$
$\rightsquigarrow y = mx$

$\therefore S = \{(1 : m : 0)\}$

This means all lines with same slope m viewed in $\mathbb{P}^2$ intersect at the same pt $(1 : m : 0)$
pt at ∞

ii) a) $C: y^2 = x^2 + 1$

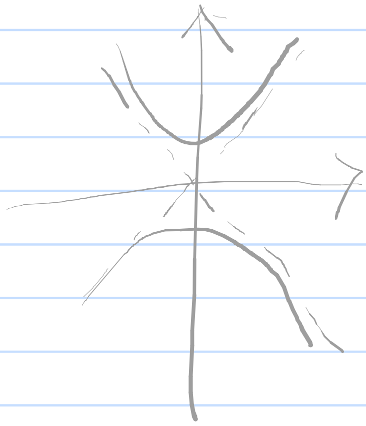
Again identify $\mathbb{A}^2 \cong U_3$

What does C corresponds to in $\mathbb{P}^2$?

Ans: similar to (i),

$(x:y:z)$ satisfying $y^2 = x^2 + z^2$, $z \neq 0$

b) Find $S = \{(x:y:z) \in \mathbb{P}^2 \mid y^2 = x^2 + z^2\} \cap H_3$ and interpret.



Ans: $z = 0 \Rightarrow x^2 = y^2$

$\therefore \{(1:1:0), (1:-1,0)\}$

$L_1: y = x$ $\qquad$ $L_2: y = -x$

The pts are where $L_1 \cap L_2 \cap C$

iii) Denote by $\tilde{J} = (\tilde{g} \mid g \in J)$, an ideal generated by the homogenisation $\tilde{g} = x_0^{\deg g}\, g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$.

x_1 first, then x_2

α, β tuple. monomial

Suppose $x^\alpha \succ x^\beta \Leftrightarrow |\alpha| > |\beta| \vee (|\alpha| = |\beta| \wedge x^\alpha > x^\beta)$

e.g. $x_1^3 x_2^2 x_3$ $\quad x = (x_1, x_2, x_3)$, $\alpha = (3, 2, 1)$

Prove that if $J = (g_1, \ldots g_r)$ is a Gröbner basis w.r.t. $\succ$, then $(\tilde{g_1}, \ldots \tilde{g_r})$ is a Gröbner basis for $\tilde{J}$ w.r.t. $\succ$, with x_0 in addition s.t. $x_1 \succ x_2 \succ \cdots \succ x_n \succ x_0$

Pf. 1. generating:

Let $h \in \tilde{J}$.

Then $h = h^1 + h^2 + \cdots + h^\alpha$ where $\alpha = \deg h$

and h^i are homo

Dehomo h^i wrt x_0, as $h^i \in \tilde{J}$,

dehomo gives $(h^i)_* = \sum_j c_{ij} g_j$

$\because$ Gröbner, $\deg g_j < \deg (h^i)_*$

$\widetilde{ag+bh} \cdot x_0^{?} \in (\tilde{g}, \tilde{h})$

$//?$

Homo, $\widetilde{(h^i)_*} \cdot x_0^k = h$

$\tilde{a}\tilde{g} + \tilde{b}\tilde{h}$

$\widetilde{c_{ij}\, g_j + \cdots} \cdot x_0^k = \sum_j c_{ij}' \, \tilde{g_j}$

$\widetilde{c_{ij} g_j} \cdot x_0^k$

$\|$

$x_0 c_{ij} \cdot G_j x_0^{k-1}$

$\|$

$c_{ij}' \cdot \tilde{g_j}$

2. $LT(\tilde{g_j}) \mid LT(\tilde{g})$ for some $\tilde{g_j}$

x_0 is the last one

the original order is 'preserved'

$xz^2 + y^2 z$

$\downarrow$ De w.r.t z

$x + y^2$

$\downarrow$ Ho

$xz + y^2$

$\downarrow \times z$

$xz^2 + y^2 z$

iv) Let $C_0 = V^{af}(x_2^2 - x_1(x_1-1)(x_1-2))$?
What is the proj extension of C_0

Ans: Homogenising by x_0 : $\deg g = 3$

$$V(x_0 x_2^2 - x_1(x_1 - x_0)(x_1 - 2x_0))$$

② Regular mapping / function

Recall :

- A quasi-proj variety is an open subset of a proj var, i.e. an intersection of a closed and an open subset
  A quasi-affine variety is an open subset of an aff var

- A regular mapping $f: V \to W$ is a function between two quasi-proj varieties s.t. $\forall P \in V$,
  $\exists$ homo poly $f_0, \dots, f_m \in \mathbb{K}^d[x_0, \dots, x_n]$,
  $f = (f_0 : \dots : f_m)$
  holds at some nbh of P in V.

- A rational mapping $f: V \dashrightarrow W$ is a class of regular mappings $f': V' \to W$ where V' is dense open subset of V, s.t. $f' \sim f'' \Leftrightarrow f' = f''$ on $V' \cap V''$.

  f is regular at $P \Leftrightarrow \exists [f]$ defined on P

  $\operatorname{dom} f = \{$all regular points of $f\}$

  f is regular $\Leftrightarrow$ f is regular at all $P \in V$

i) Let V be an irr Aff var. Show $V_h := V \setminus V(h)$ is also affine, where $h \in \mathbb{k}[V]$ is a poly function.

Pf. $V = V(f_1, \dots, f_m)$, then $\mathbb{k}[V] = \mathbb{k}[x_1, \dots, x_n] / (f_1, \dots, f_m)$

Claim: regular funct on $V \setminus V(h)$ is poly funct on $V \setminus V(h)$

Suppose $f : V \to \mathbb{k}$ is regular, U_f, U_g overlapping open subset in V.

$\exists$ $f_0, \dots, f_m$, $f = (f_0 : \dots : f_m)$ for U_f

$\exists$ $g_0, \dots, g_m$, $f = (g_0 : \dots : g_m)$ for U_g

Then at $U_f \cap U_g$, $f_i = g_i$ $\Rightarrow$ $f_i = g_i$ for $U_f \cup U_g$

Since V is irr, every 2 open subsets overlap.

$\Rightarrow \mathbb{k}[V \setminus V(h)] =$ reg funct on V_h

By Thm, $\mathbb{k}[V \setminus V(h)] = \mathbb{k}[V][h^{-1}] = \mathbb{k}[x_1, \dots, x_n, x_{n+1}] / (f_1, \dots, f_m, h x_{n+1} - 1)$

$\therefore V \setminus V(h) \cong V(f_1, \dots, f_m, h x_{n+1} - 1)$

ii) Prove that every rational mapping $f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is regular.

Pf. We show that given a rep $[f]$ regular map, $[f]$ can be extended to whole $\mathbb{P}^1$.

$$[f] : (x_0 : x_1) \mapsto (f_0(x_0 : x_1) : \dots : f_n(x_0 : x_1))$$

this is undefined $\Leftrightarrow$ $f_i(x_0 : x_1) = 0$ $\forall i$

$\Leftrightarrow$ f_i has common factor, which is a linear term

Now removing this common factor, we have another rep $[f']$ for f.

Continuing, we get a rep which is not 0 at $(x_0 : x_1)$ $\forall i$.

iii) Prove that every rational funct $\mathbb{A}^2 \dashrightarrow \mathbb{k}$, $\mathbb{k}$ alg closed which is regular on $\mathbb{A}^2 \setminus \{0\}$, is regular.

(Hint: $\mathbb{k}[x_1, x_2]$ is a UFD)

Pf. Suppose f is regular on $\mathbb{A}^2 \setminus \{0\}$.

Write $f = \dfrac{F_1(x_1, x_2)}{F_0(x_1, x_2)} : \mathbb{A}^2 \dashrightarrow \mathbb{k}$

It suffices to show that there cannot be an F_0 s.t. $F_0(0,0) = 0$ but $F_0(x_1, x_2) \neq 0$ $\forall$ $\mathbb{A}^2 \setminus \{0\}$.

Since $\mathbb{k}[x_1, x_2]$ is a UFD,
$F_0 = F_{01} \cdots F_{0j}$, prime decomp

but $V(F)$ for an irr F is infinite

iv) Prove that $\mathbb{A}^2 \setminus \{0\}$ is not an affine var

Pf. Recall $\overset{\text{iso}}{\text{poly}}$ maps $\iff$ $\mathbb{k}$ alg $\overset{\text{iso}}{\text{homo}}$ between corr coor ring

By (iii), $\mathbb{k}[\mathbb{A}^2] \cong \mathbb{k}[\mathbb{A}^2 \setminus \{0\}]$
$\Rightarrow \mathbb{A}^2 \cong \mathbb{A}^2 \setminus \{0\}$ $\lightning$

v) Prove That $\mathbb{P}^2 \setminus \{0\}$ is not an affine var

Pf. Suppose $\mathbb{P}^2 \setminus \{0\}$ is affine.

$(\mathbb{P}^2 \setminus \{0\}) \setminus H_0 \cong \mathbb{A}^2 \setminus \{0\}$
where $H_0 = V(x_0)$

Then By (i), $(\mathbb{P}^2 \setminus \{0\}) \setminus V(x_0)$ is affine, yet by (iv) $\mathbb{A}^2 \setminus \{0\}$ is not $\lightning$