Last time

Let V be an irr aff var. Show $V_h \setminus V(h)$ is also affine, where $h \in \mathbb{k}[V]$ is a poly function.

Pf. Suppose $V \setminus V(h)$ is aff, since V affine $V \cong V(I)$ with $I = (f_1, \dots, f_m)$

Claim: $V \setminus V(h) \cong V(J) \subseteq \mathbb{A}^{n+1}$

where $J = (f_1, \dots, f_m, h x_{n+1} - 1)$

Pf of claim:

Consider $\pi : V(J) \to X \setminus V(f)$

$(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n)$

and $\pi^{-1} : X \setminus V(f) \to V(J)$

$(x_1, \dots, x_n) \mapsto \left(x_1, \dots, x_n, \dfrac{1}{h(x_1, \dots, x_n)}\right)$

is bijection function

$\because$ π is regular, π^{-1} is regular, $\therefore V \setminus V(h) \cong V(J)$

Iso to aff var:

$\frac{1}{x}$ not poly in $\mathbb{A}^1$

but in $\mathbb{A}^1$

$\frac{1}{x} \cong xy = 1$

poly

① Birational Equiv

Recall :

- A rational mapping $f : V \dashrightarrow W$ is a class of regular mappings $f' : V' \to W$ where V' is dense open subset of V, s.t. $f' \sim f'' \Leftrightarrow f' = f''$ on $V' \cap V''$.

- A rational map $f : V \dashrightarrow W$ is dominant $\Leftrightarrow$ $\operatorname{im} f \subseteq W$ is dense

- A dominant rat map $f : V \dashrightarrow W$ is a birationally equiv
$\Leftrightarrow$ $\exists$ dominant rat map $g : W \dashrightarrow V$ s.t. $gf = 1$, $fg = 1$
$\Leftrightarrow$ $\mathbb{k}(V) \cong \mathbb{k}(W)$

just 1 rep

- Quasi-proj variety V is rational $\Leftrightarrow$ V is birationally equiv to $\mathbb{A}^d$ (or $\mathbb{P}^d$)

i) Hyperbola is rational.

affine: $t \mapsto (t, \frac{1}{t})$

Let $H = V(y_1 y_2 - y_0^2)$

Then
$$\mathbb{P}^1 \to H$$
$$(x_0 : x_1) \mapsto (\underset{y_0}{x_0 x_1} : \underset{y_1}{x_1^2} : \underset{y_2}{x_0^2})$$
is regular, with inverse
$$H \to \mathbb{P}^1$$
$$(y_0 : y_1 : y_2) \mapsto (y_0 : y_1)$$
$\therefore$ $H \cong \mathbb{P}^2$ and $H_0 \cong \mathbb{A}^1$

ii) Two quasi-proj var V, W are birat equiv $\Leftrightarrow$
$\exists$ open dense subsets $V' \subseteq V$, $W' \subseteq W$ s.t. $V' \cong W'$.

Pf. $(\Rightarrow)$: Trivial.
$(\Leftarrow)$: Let $f : V \dashrightarrow W$, $g : W \dashrightarrow V$ be birat equiv.
Let $V' = \operatorname{dom} f \cap f^{-1}(\operatorname{dom} g)$, $W' = \operatorname{dom} g \cap g^{-1}(\operatorname{dom} f)$

Then
$f(V') \subseteq \operatorname{dom} g$, and $gf(V') = \operatorname{id}(V') \subseteq \operatorname{dom} f \Rightarrow f(V') \subseteq W'$
Similarly, $g(W') \subseteq V'$

iii) We have a birational equiv

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

$$(x_0 : x_1 : x_2) \mapsto (x_1 x_2 : x_2 x_0 : x_0 x_1)$$

Find the open subsets $V', W' \subseteq \mathbb{P}^2$ s.t. $f|_{V'} : V' \to W'$ is iso.

Ans: Consider the open set in $\mathbb{P}^2$ $V_0 := \{(1 : x_1 : x_2)\}$ complement $x_1 = 0$

If $x_1 \neq 0$, $x_2 \neq 0$ then

$$f(V_0) = \{(1 : \frac{1}{x_1} : \frac{1}{x_2})\}$$

$$ff(V_0) = \{(\frac{1}{x_1 x_2} : \frac{1}{x_2} : \frac{1}{x_1})\}$$

$$= \{(1 : x_1 : x_2)\}$$

$\therefore$ $V' = V_0$, $W' = f(V_0)$

② Product Proj Var

i) Let X be a proj var and $f : X \to Y$ be a regular map.
Then $\operatorname{im} f \subseteq Y$ is closed

write $f = (f_0 : \cdots f_m)$

Pf. Γ_f is closed in $X \times Y$ since Γ_f contains
$((x_1 : \ldots : x_n), (y_1 : \ldots ; y_m)) \in X \times Y$
for which locally, $y_j = A f_j(x_1 : \cdots : x_n)$ (multiple of f)
$\Leftrightarrow \quad y_j f_k() = y_k f_j()$
which is a polynomial
$\therefore \Gamma_f$ closed
$\because \pi$ is closed, $\operatorname{im} f = \pi(\Gamma_f)$ is closed

ii) Let X be a projective var.
Show that X is affine $\Leftrightarrow$ X is finite

Pf. WLOG, assume X is irr
$f : X \hookrightarrow \mathbb{A}^n$, $f = (f_1 : \cdots : f_n)$
Thm. every reg funct $f : X \to k$ on an irr proj var X is const
so each f_i is const $\Rightarrow f$ is const $\Rightarrow X$ is $\{*\}$
For reducible, $X = \underbrace{X_1 \cup \cdots \cup X_n}_{\text{fin}}$

iii) E.g. $\mathbb{A}^n$ is projective $\Leftrightarrow$ $n = 0$

iv) Show $\mathbb{A}^2 \setminus \{0\}$ is not projective

Ans: Thm. reg funct $f : X \to k$ on irr proj var X is const

$\mathbb{A}^2 \setminus \{0\}$ is irr

$\rightsquigarrow$ Find a reg funct $f : \mathbb{A}^2 \setminus \{0\} \to k$ not const but reg
$(x, y) \mapsto x$ (π)

vi) Show $\mathbb{P}^n \times \mathbb{P}^m$ is birat equiv $\mathbb{P}^{n+m}$

$\mathbb{A}^n \cong U_i$

Ans: $\mathbb{P}^n \simeq_{b.e.} \mathbb{A}^n$

$$\mathbb{P}^n \times \mathbb{P}^m \simeq_{b.e.} \mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m} \simeq_{b.e.} \mathbb{P}^{n+m}$$

vii) Show $\mathbb{P}^1 \times \mathbb{A}^1$ is not affine nor proj

Ans: Not affine: $\mathbb{P}^1 \times \{0\} \subseteq \mathbb{P}^1 \times \mathbb{A}^1$

If $\mathbb{P}^1 \times \mathbb{A}^1$ is affine, $\mathbb{P}^1 \times \mathbb{A}^1$ can be embedded into $\mathbb{A}^n$

$\Rightarrow \mathbb{P}^1 \times \{0\}$ embedded into $\mathbb{A}^n$, $\lightning$

Not proj: $\{0\} \times \mathbb{A}^1$ is closed in $\mathbb{P}^1 \times \mathbb{A}^1$

If $\mathbb{P}^1 \times \mathbb{A}^1$ is embedded into $\mathbb{P}^n$ as a closed subset,

then $\underset{\underset{U}{\cup|}}{\mathbb{A}^1}$ is closed in $\mathbb{P}^n$. $\lightning$