⓪ Segre embedding - example

i) Show that the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ has image $V(y_0 y_3 - y_1 y_2)$

Ans: $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^3$

$$((x_0 : x_1), (x_2 : x_3)) \mapsto (\underset{y_0}{x_0 x_2} : \underset{y_1}{x_0 x_3} : \underset{y_2}{x_1 x_2} : \underset{y_3}{x_1 x_3})$$

which is $V(y_0 y_3 - y_1 y_2)$

ii) Consider $\mathbb{P}^3 \supseteq V(y_0 y_3 - y_1 y_2) \xrightarrow{f^{-1}} \mathbb{P}^1 \times \mathbb{P}^1$

$$(y_0 : y_1 : y_2 : y_3) \mapsto ((y_0 : y_2), (y_0 : y_1))$$

or $((y_1 : y_3), (y_2 : y_3))$

each not defined everywhere

$$y_0 y_0 : y_0 y_1 : y_2 y_0 : y_2 y_1$$

$y_0 y_3$

Find out image of $\pi_1 f^{-1}$ & $\pi_2 f^{-1}$.

Ans: $\pi_1 f^{-1}$: $V(y_0 y_3 - y_1 y_2) \xrightarrow{f^{-1}} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\pi_1} \mathbb{P}^1$

$$(y_0 : y_1 : y_2 : y_3) \longmapsto (y_0 : y_2) \text{ or } (y_1 : y_3)$$

$\pi_2 f^{-1}$: $V(y_0 y_3 - y_1 y_2) \xrightarrow{f^{-1}} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\pi_2} \mathbb{P}^1$

$$(y_0 : y_1 : y_2 : y_3) \longmapsto (y_0 : y_1) \text{ or } (y_2 : y_3)$$

① Veronese variety

The Veronese surface is the image of

$$v : \mathbb{P}^2 \to \mathbb{P}^5$$
$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_1 x_2 : x_0 x_2 : x_0 x_1)$$

i) Show that the image of a line in $\mathbb{P}^2$ under v is contained in a conic in $\mathbb{P}^5$

Ans: By appropriate change of coord, WLOG, let the line be represented by $x_2 = 0$.

Now

$$(x_0 : x_1 : 0) \mapsto (x_0^2 : x_1^2 : 0 : 0 : 0 : x_0 x_1)$$

Denote by $(y_0 : y_1 : y_2 : y_3 : y_4 : y_5)$ the coor in $\mathbb{P}^5$.

Then the image is contained in the variety defined by $y_5^2 - y_0 y_1 = 0$, which is a conic.

ii) Show that the image of a conic in $\mathbb{P}^2$ under v is contained in a quartic in $\mathbb{P}^5$.

completing sq

Ans: By appropriate change of coord, WLOG, let the conic be represented by

$$x_0^2 - (b x_1^2 + c x_2^2) = 0$$

So a point on the conic has the form

$$(\pm \sqrt{b x_1^2 + c x_2^2} : x_1 : x_2)$$

which is mapped to

$$(b x_1^2 + c x_2^2 : x_1^2 : x_2^2 : x_1 x_2 : x_2 \sqrt{b x_1^2 + c x_2^2} : x_1 \sqrt{b x_1^2 + c x_2^2})$$

or $(b x_1^2 + c x_2^2 : x_1^2 : x_2^2 : x_1 x_2 : -x_2 \sqrt{b x_1^2 + c x_2^2} : -x_1 \sqrt{b x_1^2 + c x_2^2})$

which is contained in the variety defined by

$$y_1 y_2 y_4 y_5 - y_0 y_3^3 = 0,$$

which is a quartic

$y_1 y_2 y_4 y_5$

$x_1 x_2 (b x_1^2 + c x_0^2) x_1^2 x_2^2$

$= b x_1^5 x_2^3 + c x_1^3 x_2^5$

$y_0 y_3^3$

② Grassmannian as a proj space

Def. $G(k, n)$ is the set of all k-dim subspace in the vector space $\mathbb{K}^n$, where $\mathbb{K}$ is a field.
$V(k,n)$ is the set of sets of k-lin indep vectors in $\mathbb{K}^n$.

i) Define

$$\gamma : V(k,n) \longrightarrow G(k,n)$$
$$(v_1, \cdots, v_k) \mapsto [v_1, \cdots, v_k]$$

where $[v_1, \cdots, v_k]$ denotes the span of v_i.
So an element in $V(k,n)$ defines an element in $G(k,n)$
Is γ injective?

Ans: No. A subspace, by change of basis, can have different bases.

ii) $G(k,n)$ can be seen as a projective space via the Plücker embedding:

$\mathbb{P}(V) = (V \setminus \{0\})/\sim$, $u \sim v \Leftrightarrow \exists\, k \in \mathbb{K}\setminus\{0\}, v = ku$

$G(\otimes \mathcal{I})$

$\Lambda(V) = T(V)/I$

vector subspace of $\Lambda(V)$

$x_1 \wedge \cdots \wedge x_k$

$$i : G(k,n) \to \mathbb{P}(\Lambda^k \mathbb{K}^n)$$
$$[v_1, \cdots, v_k] \mapsto [v_1 \wedge \cdots \wedge v_k]$$

where $\Lambda^k \mathbb{K}^n$ is the k^{th} exterior power of $\mathbb{K}^n$

Show that i is well-defined, i.e. $[v_1, \cdots, v_k] = [w_1, \dots, w_k]$
$\Rightarrow i(v_1, \cdots, v_k) = i(w_1, \dots, w_k)$

Ans: If $[v_1, \cdots, v_k] = [w_1, \cdots, w_k]$,
then there is a change of basis matrix $Q = (q_{ij})$
s.t. $w_i = \sum_{j=1}^{k} q_{ij} v_j$

$\Rightarrow$ $w_1 \wedge \cdots \wedge w_k = \det(Q)\, v_1 \wedge \cdots \wedge v_k$
$\rightsquigarrow$ differ by a scalar multiple
$\Rightarrow$ proj

(3) Dimension of varieties – Trancendence degree

Let K be a fin gen field ext of $\mathbb{k}$.

The Transcendence deg of K over $\mathbb{k}$, $\operatorname{tr\,deg}_{\mathbb{k}} K$, is defined to be the smallest integer n s.t. for $x_1, \ldots, x_n \in K$, K is algebraic over $\mathbb{k}(x_1, \ldots, x_n)$, i.e. every element of K is a root of some non-zero poly with coeff in $\mathbb{k}(x_1, \ldots, x_n)$.

i) $\operatorname{tr\,deg}_{\mathbb{k}} K$ is well-defined:

Let X be a variety, and $\mathbb{k}(X)$ be the field of rational functions.

Prove that for two maximal system of algebraically independent elements, $\{a_1, \ldots, a_s\}$ and $\{b_1, \ldots, b_t\}$ in $\mathbb{k}(X)$, we have

$$s = t \quad \text{and} \quad \mathbb{k}(a_1, \ldots, a_s) \cong \mathbb{k}(b_1, \ldots, b_t)$$

Pf. Each b_i is a root of polynomials with coeff in $\mathbb{k}(a_1, \ldots, a_s)$

WLOG, suppose the polynomial satisfied by b_1 contains a_1.

Then $\{b_1, a_2, \ldots, a_s\}$ is another maximal system.

We continue inductively.

If $t > s$, then $\{b_1, \ldots, b_s\}$ is another maximal system, contradicting to the alg indep of $b_{s+1}, \ldots, b_t$.

If $t < s$, then $\{b_1, \ldots, b_t, a \ldots\}$ is another maximal system, contradicting to the maximality of $\{b, \ldots, b_t\}$.

$\therefore s = t$

Now $\{a_1, \ldots, a_s\}$, $\{b_1, \ldots, b_s\}$ are the 2 max systems.

Let V be a variety (aff / proj).

$f_1 \notin I(V)$ $\frac{f_1}{f_0} \neq 0$ has inverse $\frac{f_0}{f_1}$ then $\frac{f_1}{f_0} \in \mathbb{k}(V)$

$\mathbb{k}(V)$ is a fin gen field ext of $\mathbb{k}$ $\to \mathbb{k}$ (fin many)

The dim of V, $\dim(V)$ can be defined as $\mathrm{Tr\,Deg}_{\mathbb{k}}\, \mathbb{k}(V)$. $\mapsto$ many equiv way to define

ii) Show the following: Consider irreducible varieties.

1. If U is an open dense subvariety of V, $\dim(U) = \dim(V)$
2. If $V = V(F)$, define $V^* = V(F^*)$. Then $\dim(V) = \dim(V^*)$
3. $\dim(V) = 0 \iff V = \{x\}$
4. Let V be a closed subvariety of $\mathbb{A}^2$ (resp. $\mathbb{P}^2$).
   $\dim(V) = 1 \iff V$ is an affine (resp. proj) plane curve.

Pf. (1), (2): $\mathbb{k}(U) \cong \mathbb{k}(V)$

$\mathbb{k}(V^*) \cong \mathbb{k}(V)$ $\quad \frac{f}{g} \leadsto \frac{f^d}{g^d}$ homogenise mult by x?

(3): By (2), suppose V is affine.

$\dim V = 0 \iff \mathbb{k}(V)$ is algebraic over $\mathbb{k}$

$\frac{f}{g} = $ const in $\mathbb{k}(V)$ $\leadsto \iff \mathbb{k}(V) = \mathbb{k}$ since $\mathbb{k}$ is alg closed

$\iff \mathbb{k}[V] = \mathbb{k}$

In previous tutorial, we showed that $\exists F$ s.t. $F(p_1) = 1$ but $F(p_j) = 0$ $\forall j$

Since constant functions do not separate pts, this means V has only 1 pt.

(4): For $\mathbb{A}^2$, let $\mathbb{k}(V) \subseteq \mathbb{k}(x,y)$

Since $\dim(V) = 1$, WLOG, x is transcendent over $\mathbb{k}$, and y is algebraic over $\mathbb{k}(x)$ (in $\mathbb{k}(V)$).

Thus y is the root of a poly f with coeff in $\mathbb{k}(x)$.

$V = V(f)$ is a plane curve.

For $\mathbb{P}^2$, use (2).

④ Tangent space and (non-) singular curve *Affine*

Def. Let $I \subseteq \Bbbk[x_1, \dots, x_n]$.

$$I^{(1)}_P := \{ df(P) \mid f \in I \} \subseteq \Bbbk^{(1)}[x_1, \dots, x_n]$$

where $df(P) = \frac{\partial f(P)}{\partial x_0} dx_0 + \dots + \frac{\partial f(P)}{\partial x_n} dx_n$, *we may identify dx_i with x_i*

is called the linear part of I at P.

Let X be an irr affine var.

$$T_P X = \{ v \in \Bbbk^n \mid \forall \alpha \in I(X)^{(1)}_P : \alpha(v) = 0 \}$$ *substitute v_i into x_i*

A point $P \in X$ is called non-singular (smooth) $\Leftrightarrow$ $\dim T_P X = \dim(X)$.

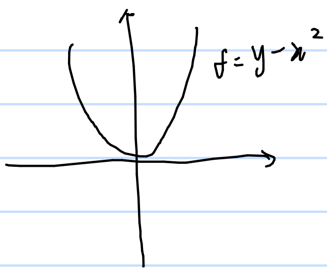
Rk. In $\mathbb{A}^2$, a curve has dim 1, the ambient space has dim 2.

$\dim T_P X = 2 \Leftrightarrow df(P) = (0,0)$

$\therefore$ P is non-singular (smooth) $\Leftrightarrow$ $df(P) \neq (0,0)$

A curve is called non-singular (smooth) $\Leftrightarrow$ all P's on it are non-singular (smooth)

E.g. 1

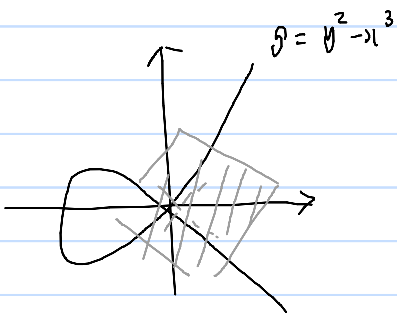


$\frac{\partial f}{\partial x} = -2x$, $\frac{\partial f}{\partial y} = 1$

$df(P) \neq (0,0) \ \forall P$

$\therefore$ non-singular

E.g. 2



$\frac{\partial g}{\partial x} = -3x^2$, $\frac{\partial g}{\partial y} = 2y$

$df(0,0) = (0,0)$

$\therefore$ singular at P