DÚ 1. Denote by $A[\![x]\!]$ the ring of formal power series with coefficients in A, i.e. its elements are formal expressions

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum a_n x^n.$$

1. Let $J \subseteq R$ be an ideal and define the limit of the diagram

$$\cdots \to R/J^2 \to R/J^2 \to R/J$$

   to be the completion of R at the ideal J and denote it $\widehat{R}_J$. Show that $A[\![x]\!] = \widehat{A[x]}_{\mathfrak{m}_0}$, the completion of the polynomial ring at the maximal ideal $\mathfrak{m}_0 = (x)$. Consider all the rings in the diagram equipped with the discrete topology and endow $A[\![x]\!]$ with the limit topology. Give an explicit criterion for a sequence (or a net or a filter) of formal power series $f_k(x) = \sum a_{kn} x^n$ to converge to $f(x) = \sum a_n x^n$ in terms of the coefficients a_{kn} and a_n.

2. Show that for a field $\Bbbk$ the ring $\Bbbk[\![x]\!]$ is a UFD. As one of the ingredients, prove more generally that $f(x) \in A[\![x]\!]$ is invertible iff $a_0 \in A^\times$ is invertible; in the nontrivial direction, reduce to the case $a_0 = 1$, write $f(x) = 1 + g(x)$ so that $g(x) = a_1 x + a_2 x^2 + \cdots$, and show that the following makes sense and defines the inverse:

$$f(x)^{-1} = (1 + g(x))^{-1} = 1 - g(x) + g(x)^2 - \cdots$$

   (using the previous point, one can make sense of the infinite sum on the right hand side as the limit of the sequence of partial sums; this indeed converges).

3. The goal is to prove that $\Bbbk[\![\mathbb{x}]\!]$ is a UFD. As for the polynomials, one observes that for a tuple of variables $\mathbb{x} = (\mathbb{x}', t)$, one gets $\Bbbk[\![\mathbb{x}]\!] = \Bbbk[\![\mathbb{x}']\!][\![t]\!]$. However, it is generally not true that A being a UFD implies $A[\![x]\!]$ being a UFD, so one has to argue differently. We say that $f(\mathbb{x}) \in \Bbbk[\![\mathbb{x}]\!]$ is a **not-necessarily-monic** Weierstrass polynomial w.r.t. t if it lies in

$$\Bbbk[\![\mathbb{x}']\!][t] \subseteq \Bbbk[\![\mathbb{x}']\!][\![t]\!]$$

   and as such is a polynomial of degree n with leading coefficient invertible in $\Bbbk[\![\mathbb{x}']\!]$ and all other coefficients non-invertible, i.e. lying in $\mathfrak{m}_0[\![t]\!]$. Classically, a Weierstrass polynomial is additionally assumed to be monic. The Weierstrass preparation theorem says that every formal power series $f(\mathbb{x}) \in \Bbbk[\![\mathbb{x}']\!][\![t]\!] \smallsetminus \mathfrak{m}_0[\![t]\!]$ is associated to a (unique monic) Weierstrass polynomial (the condition simply means that f is non-zero along the t-axis, i.e. that as an element of $\Bbbk[\![\mathbb{x}]\!]$ it contains some monomial t^n with a non-zero coefficient).[1] Since we can assume that $\Bbbk[\![\mathbb{x}']\!][t]$ is a UFD by induction, this can be

[1] Here is an idea of the proof: We want $f(\mathbb{x}) = a(\mathbb{x})(t^n + b(\mathbb{x}))$ with $b(\mathbb{x}) \in \mathfrak{m}_0[t]$ of degree $n-1$. Now rewriting this as $t^n = a(\mathbb{x})^{-1} f(\mathbb{x}) - b(\mathbb{x})$ we want to divide t^n by $f(\mathbb{x})$ "with a remainder". Now one can easily divide by t^n, for one can write canonically any formal power series as $a_0(\mathbb{x})t^n + b_0(\mathbb{x})$ with $b_0(\mathbb{x}) \in \Bbbk[\![\mathbb{x}']\!][t]$ of degree $n-1$. Expressing $f(\mathbb{x})$ in this way as $f(\mathbb{x}) = u(\mathbb{x})t^n + v(\mathbb{x})$, now with $v(\mathbb{x}) \in \mathfrak{m}_0[t]$ – this determines n, we see that $f(\mathbb{x})$ is roughly $u(\mathbb{x})t^n$ and we start by dividing by $u(\mathbb{x})t^n$ instead:

$$a_0(\mathbb{x})t^n + b_0(\mathbb{x}) = a_0(\mathbb{x})u(\mathbb{x})^{-1} \underbrace{u(\mathbb{x})t^n}_{f(\mathbb{x})-v(\mathbb{x})} + b_0(\mathbb{x}) = a_0(\mathbb{x})u(\mathbb{x})^{-1} f(\mathbb{x}) + (\underbrace{-a_0(\mathbb{x})u(\mathbb{x})^{-1}v(\mathbb{x})}_{a_1(\mathbb{x})t^n + b_1(\mathbb{x})} + b_0(\mathbb{x})).$$

Continuing in this way we see that the quotient is $(a_0(\mathbb{x}) + a_1(\mathbb{x}) + \cdots)u(\mathbb{x})^{-1}$ and the remainder is $b_0(\mathbb{x}) + b_1(\mathbb{x}) + \cdots$ provided that these converge. But one can see easily, using $v(\mathbb{x}) \in \mathfrak{m}_0[t]$, that if $a_k(\mathbb{x}) \in \mathfrak{m}_0^k[\![t]\!]$ then both $a_{k+1}(\mathbb{x}), b_{k+1}(\mathbb{x}) \in \mathfrak{m}_0^{k+1}[\![t]\!]$.

ultimately used to show that $\Bbbk[\![\mathbb{x}]\!]$ is a UFD: Assume that $M \subseteq N$ are two (cancellative) monoids satisfying:

- The mapping $M/\text{ass} \to N/\text{ass}$ is bijective. In detail, every element of N is associated to some element of M and every two elements of M that are associated in N are associated also in M.

Show that if M has a unique factorization property then so does N. Apply this to M consisting of the (**not-necessarily-monic**) Weierstrass polynomials (these easily inherit the unique factorization property from $\Bbbk[\![\mathbb{x}']\!][t]$ for they are closed both under multiplication and factorization) and $N = \Bbbk[\![\mathbb{x}']\!][\![t]\!] \smallsetminus \mathfrak{m}_0[\![t]\!]$. Finally, show that for any non-zero element $f(\mathbb{x}) \in \Bbbk[\![\mathbb{x}]\!]$ one can set up the coordinates so that $f(\mathbb{x}) \in N$ (look at the lowest non-zero homogeneous degree and apply the theorem from algebraic geometry about polynomials).

DÚ 2. Here we work over $\mathbb{R}$ (or better over $\mathbb{C}$). Let $C = V(f) \subseteq \mathbb{A}^2$ be a curve that is smooth at $\mathbb{x}_0 \in C$ in the sense that $\mathrm{d}f(\mathbb{x}) \neq 0$. Then one can parametrize C near $\mathbb{x}_0$ locally as $\mathbb{x} = (\xi(t), \alpha(t))$, passing through $\mathbb{x}_0$ for $t = t_0$, and any two parametrizations differ by a diffeomorphism of the parameter space near t_0, so the following definition makes sense: The multiplicity of the intersection point $\mathbb{x}_0$ of $C = V(f)$ and $D = V(g)$ is defined as the order of zero at t_0 of the function $g(\xi(t), \alpha(t))$ (here the function g is defined uniquely up to a constant if it is required to be polynomial – since it is the generator of $I(D)$ – and up to a multiple by a nonzero smooth function otherwise). It is not obvious why this definition is symmetric in C and D; you may try to give it a thought.

1. Take some concrete example like $f = x^2 + y^2 - 1$ and $g = (x - \sqrt{2})^2 + (y - \sqrt{2})^2 - 1$ and use the parametrization using sin and cos or any other to compute the multiplicity.

2. Now assume that the coordinate system has been set up so that f has a non-zero coefficient at y^r, the x-coordinates of all intersection points are distinct and also distinct from all the points of C for which $f'_x = 0$ (these form the set $V(f, f'_x)$ which must be finite unless $(x - a) \mid f$ – in other words, we want all the linear factors of f depend on y). Then near each x-coordinate of an intersection point, one can write the branches of $V(f)$ as $y = \alpha_i(x)$, i.e. the local parametrizations as above are $(x, \alpha_i(x))$. Plug the above into the formula

$$\operatorname{Res}(f, g; y) = a_r^s \cdot g(x, \alpha_1(x)) \cdots g(x, \alpha_r(x))$$

   and prove that the number of intersections is bounded by rs even when counted with multiplicity in the sense of this exercise. In the concrete example above, plugging in the non-polynomial parametrization into the formula must give a polynomial expression (the resultant is a polynomial after all), try it.

DÚ 3. Let $f\colon X \to Y$ be a regular map and denote by $\varphi = f^*\colon \Bbbk[Y] \to \Bbbk[X]$ the induced map on coordinate $\Bbbk$-algebras. We showed that $V(\varphi_* I) = f^{-1}(V(I))$.

1. Prove symmetrically that

$$I(f(S)) = \varphi^* I(S)$$

(this should be quite obvious) and deduce that $\overline{f(S)} = V(\varphi^*(I(S))$. Now specialize to $S = V(J)$ and use Hilbert Nullstellensatz to prove that

$$\overline{f(V(J))} = V(\varphi^* J).$$

(I don't know if Hilbert Nullstellensatz is really necessary – you may try to figure it out.)

2. Now assume in addition that f is dominant, i.e. that φ is injective. We showed (using Nakayama lemma) that if φ is finite then f is surjective and closed. Prove that fibres of f are all finite (this does not use the previous point, and it should have been done in the tutorial).

3. You may know the following theorem: Let A be a finitely generated $\mathbb{k}$-algebra, acted upon by a finite group G of $\mathbb{k}$-algebra automorphisms, from the right and denoted as a^g. If $n = |G|$ is not divisible by $\operatorname{char} \mathbb{k}$ then the subalgebra

$$A^G = \{a \in A \mid a^g = a\}$$

of invariants (or fixed points) is also finitely generated. In addition, A is a finite A^G-algebra.[2]

4. Assume now that an affine variety X has a (left) action of a finite group G by polynomial maps. This induces a (right) action of G on $\mathbb{k}[X]$ and we may apply the previous point to conclude that there exists an affine variety X_G such that $\mathbb{k}[X_G] = \mathbb{k}[X]^G$. Show that the canonical map $X \to X_G$ is surjective with finite fibres and identify them as orbits of the G-action. Conclude that $X_G = X/G$ as sets.

[2] Start by observing that any $a \in A$ is a root of a monic polynomial $p_a = \prod_{g \in G}(x - a^g)$ whose coefficients are the elementary symmetric polynomials $\sigma_i(a) = \sigma_i\{a^g \mid g \in G\}$, up to a sign, and as such lie in A^G. Denote the generators of A as u_j and denote by B the subalgebra generated by $\sigma_i(u_j)$. Then $p_{u_j}(u_j) = 0$ implies that $u_j^n \in B\{1, u_j, \dots, u_j^{n-1}\}$. For this reason,

$$A = B\{u_1^{\alpha_1} \dots u_r^{\alpha_r} \mid \alpha_1, \dots, \alpha_r < n\} = B\{u^\alpha\},$$

so A is already finite over B. Explicitly, for any $a \in A$, we have an expression $a = \sum b_\alpha u^\alpha$ with $b_\alpha \in B$. Apply the symmetrization $S\colon A \to A^G$, $S(a) = \frac{1}{n} \sum_{g \in G} a^g$; assuming that $a \in A^G$, we get

$$a = S(a) = \sum b_\alpha S(u^\alpha)$$

and thus $A^G = B\{S(u^\alpha)\} = \mathbb{k}[\sigma_i(u_j), S(a^\alpha)]$.

DÚ 1. ~~Dokažte následující izomorfismy:~~

- ~~$A[a^{-1}] \cong A[t]/(at-1)$,~~
- ~~$(A/I)[t] \cong A[t]/J$ a popište ideál J,~~
- ~~$A/(I+J) \cong (A/I)/J'$ a popište ideál J' ve stylu "je to v zásadě J, jenom…".~~

DÚ 2. ~~Pomocí Gröbnerovy báze vyřešte soustavu polynomiálních rovnic~~

$$\begin{aligned} x^2 + y + z &= 1 \\ x + y^2 + z &= 1 \\ x + y + z^2 &= 1 \end{aligned}$$

DÚ 3. ~~Nechť $\Bbbk$ je algebraicky uzavřené těleso. Studujte vztah mezi nenulovými kvadratickými polynomy $f \in \Bbbk[x_1, \dots, x_n]$ a příslušnými afinními varietami $V(f) \subseteq \mathbb{A}^n$; konkrétně se zabývejte tím, nakolik je zobrazení $f \mapsto V(f)$ injektivní. Dále proveďte analogickou studii pro kubické polynomy.~~

DÚ 4. Dokažte následující tvrzení:

- Afinní varieta X je ireducibilní, právě když pro libovolné afinní variety X_1, X_2 platí
$$X \subseteq X_1 \cup X_2 \implies (X \subseteq X_1 \vee X \subseteq X_2).$$
- Ideál J je prvoideál, právě když pro libovolné ideály J_1, J_2 platí
$$J \supseteq J_1 J_2 \implies (J \supseteq J_1 \vee J \supseteq J_2).$$
- Pomocí předchozích dvou tvrzení dokažte, že X je ireducibilní, právě když $I(X)$ je prvoideál (není k tomu potřeba Hilbertova věta o nulách, ale klidně ji použijte).

DÚ 5. Označme $\widetilde{I} = (\widetilde{g} \mid g \in I)$ ideál generovaný homogenizacemi $\widetilde{g} = x_0^{\deg g} g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$. (Platí $\overline{V^{\mathrm{af}}(I)} = V^{\mathrm{pr}}(\widetilde{I})$, to ukážeme na příští přednášce; toto tvrzení je důvodem, proč se tímto ideálem zabýváme.) Uvažujme následující uspořádání monomů

$$x^\alpha >_{\mathrm{gr}} x^\beta \quad \Leftrightarrow \quad |\alpha| > |\beta| \vee (|\alpha| = |\beta| \wedge x^\alpha > x^\beta).$$

Dokažte, že v případě, že $I = (g_1, \dots, g_r)$ je Gröbnerova báze vzhledem k $>_{\mathrm{gr}}$, je také $\widetilde{I} = (\widetilde{g}_1, \dots, \widetilde{g}_r)$ Gröbnerovou bází vzhledem k podobném uspořádání $>_{\mathrm{gr}}$, jen s x_0 navíc a menším než zbylé proměnné, tj. $x_1 > \dots > x_n > x_0$.

DÚ 6. Ukažte, že zobrazení $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $(x_0 : x_1 : x_2) \mapsto (x_1 x_2 : x_2 x_0 : x_0 x_1)$ je biracionální ekvivalence a najděte otevřené podmnožiny $\mathbb{P}^2$, na nichž je f izomorfismus. (Nápověda: napíšete-li si zobrazení afinně, inverze by měla být jasná.)

DÚ 7. Dokažte, že obraz regulárního zobrazení $\mathbb{A}^1 \to \mathbb{A}^n$ je uzavřený (nápověda: použijte, že $\mathbb{P}^1 \to \mathbb{P}^n$ je regulární a zkoumejte obraz nevlastního bodu).

DÚ 8. Řekneme, že k-rovina K a l-rovina L se protínají *transverzálně* v $\mathbb{P}^n$, jestliže jejich průnik je $(k+l-n)$-rovina. Ukažte, že obecná dvojice $(K, L) \in \mathbb{G}(k,n) \times \mathbb{G}(l,n)$ se protíná transverzálně.

DÚ 9. ~~Nechť $f\colon X \to Y$ je *surjektivní* uzavřené zobrazení mezi Noetherovskými topologickými prostory takové, že pro každou dvojici uzavřených podmnožin $A \subsetneqq B \subseteq X$, kde B je ireducibilní, je $f(A) \subsetneqq f(B)$. Dokažte, že f je otevřené.~~