10.1 $f(x) = \overline{\sum_{n=1}^{\infty}} n e^{-nx}$

Vrsta stojesmo ne konvergira do Weierstrassova kriterija na intervalu $[\ln 2, \ln 3]$, ne $\bar{\text{toč}}$.:

$\underset{\| \; n(e^x)^{-n}}{n e^{-nx}} \underline{\leq} h(n)$ na $[\ln 2, \ln 3]$

t. d. $\overline{\sum} h(n)$ konvergira

$x = \ln 2 \rightsquigarrow n e^{-nx} = n 2^{-n} = \frac{n}{2^n}$

$x = \ln 3 \rightsquigarrow n e^{-nx} = n 3^{-n} = \frac{n}{3^n}$

tedy $\boxed{n e^{-nx} \leq \frac{n}{2^n} = h(n)}$

pro $x \in [\ln 2, \ln 3]$

$\sum \frac{n}{2^n}$ konverguje dle podílového kritéria

$$\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{n} \cdot \frac{1}{2} \longrightarrow \frac{1}{2} < 1$$

pro $n \to \infty$

$$\int_{\ln 2}^{\ln 3} f(x) = \int_{\ln 2}^{\ln 3} \left(\sum_{n=1}^{\infty} n e^{-nx} \right) dx$$

$$= \sum_{n=1}^{\infty} \left(\int_{\ln 2}^{\ln 3} n e^{-nx} dx \right) =$$

$$= \sum_{n=1}^{\infty} n \left[-\frac{1}{n} e^{-nx} \right]_{\ln 2}^{\ln 3} =$$

$$= -\sum_{n=1}^{\infty} \left(3^{-n} - 2^{-n} \right) =$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n} \right) =$$

$$= \sum_{1}^{\infty} \frac{1}{2^n} - \sum_{1}^{\infty} \frac{1}{3^n}$$

$$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} - \frac{1}{3} \frac{1}{1-\frac{1}{3}} =$$

$$= \frac{1}{1} - \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$$

10.2 (i) $y' = \frac{\sqrt{1-y^2}}{\cos^2 x}(1+\cos^2 x)$

$$\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = \frac{1+\cos^2 x}{\cos^2 x} \quad / \cdot dx$$

$$\frac{1}{\sqrt{1-y^2}}\,dy = \frac{1+\cos^2 x}{\cos^2 x}\cdot dx \;/\int$$

"separowane zmienne"

$$\int \frac{dy}{\sqrt{1-y^2}}\,dy = \int \left(\frac{1}{\cos^2 x}+1\right)dx$$

$$\arcsin y = \operatorname{tg} x + x + C$$

$$\boxed{y = \sin(\operatorname{tg} x + x + C) \quad C \in \mathbb{R} \qquad y = \pm 1}$$

$\cos^2 x \neq 0 \qquad \cos x \neq 0$

$$\boxed{x \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}}$$

$\operatorname{tg} x$ jest ~~zdefiniowany~~ zdefiniowany

(ii) $y' = \dfrac{y^2+1}{x+1}$ $t \geq 0$. $y(0) = 1$

$$\frac{1}{y^2+1} \cdot \frac{dy}{dx} = \frac{1}{x+1}$$

$$\int \frac{1}{y^2+1}\, dy = \int \frac{1}{x+1}\, dx$$

$$\operatorname{arctg} y = \ln|x+1| + C$$

$$\boxed{y = \operatorname{tg}(\ln|x+1| + C) \quad C \in \mathbb{R}}$$

$$y(0) = 1 \rightsquigarrow 1 = \operatorname{tg}(0 + C)$$

$$1 = \operatorname{tg} C \Rightarrow$$

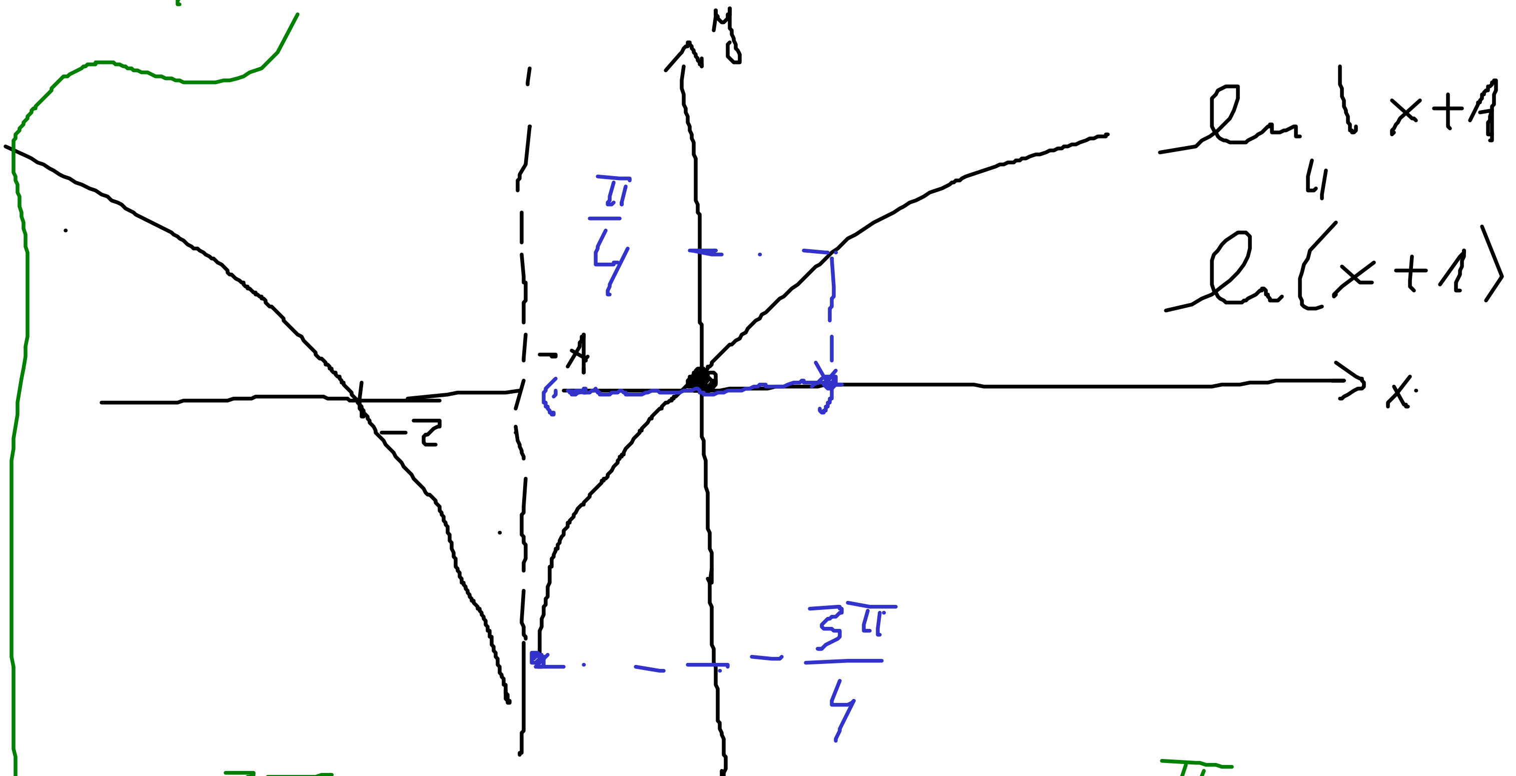
$$\Rightarrow \boxed{C \in \frac{\pi}{4} + k\pi}$$

$$\boxed{C=\frac{\pi}{4},\quad y(x)=\operatorname{tg}\left(\ln|x+1|+\frac{\pi}{4}\right)}$$

$x \neq -1$

$$-\frac{\pi}{2} < \ln|x+1| + \frac{\pi}{4} < \frac{\pi}{2}$$

$$-\frac{3\pi}{4} < \ln|x+1| < \frac{\pi}{4}$$



$$e^{-\frac{3\pi}{4}} < x+1 < e^{\frac{\pi}{4}}$$

$$\boxed{x \in \left(e^{-\frac{3\pi}{4}}-1,\ e^{\frac{\pi}{4}}-1\right)}$$

10.3 funkce $[1,2] \to \mathbb{R}$

tvoří nekonečně dim.

vektorový prostor

$f, g : [1,2] \to \mathbb{R}$

$\int_1^2 (f \cdot g) dx = \langle f, g \rangle$

skalární součin

Uvažme 2-dim podprostor

$\underbrace{\langle x^2, \frac{1}{x} \rangle}$ $= V$ lin. obal

podprostor generovaný funkcemi x^2, $\frac{1}{x}$

(1) Najdeme ON-bázi (f, g) prostoru V

$f := f = \frac{1}{x}$

Nejprve najdeme OG bázi $\left(f=\frac{1}{x}, h\right)$, kde $h = x^2 + a\frac{1}{x}$

$a \in \mathbb{R}$

Sk. součin $\langle f, h\rangle = \int_1^2 \frac{1}{x}\left(x^2 + a\frac{1}{x}\right)dx$

$$= \int_1^2 \left(x + a\frac{1}{x^2}\right)dx =$$

$$= \left[\frac{1}{2}x^2 - a\frac{1}{x}\right]_1^2 =$$

$$= \frac{1}{2}\left(2^2 - 1^2\right) - a\left(\frac{1}{2} - \frac{1}{1}\right) =$$

$$= \frac{3}{2} - a\left(-\frac{1}{2}\right) = \frac{3}{2} + \frac{a}{2} = 0$$

$$\Rightarrow \underline{\underline{a = -3}}$$

OG báze $\left(f = \frac{1}{x}, h = x^2 - \frac{3}{x}\right)$

- ověříme, že $\left\langle \frac{1}{x}, \frac{1}{x}\right\rangle = \int_1^2 \frac{1}{x^2}dx$

$$= \left[-\frac{1}{x}\right]_1^2 = \left[-\frac{1}{2} - (-1)\right] = \frac{1}{2}$$

$$\left\| \frac{1}{x} \right\| = \sqrt{\left\langle \frac{1}{x}, \frac{1}{x} \right\rangle} = \frac{1}{\sqrt{2}}$$

OG - bázom mám stačí

<u>Pozn.:</u> $\sqrt{2} \cdot \frac{1}{x}$ má dĺžku 1

- $$\left\langle x^2 - \frac{3}{x}, x^2 - \frac{3}{x} \right\rangle = \int_1^2 \left(x^2 - \frac{3}{x}\right)^2 dx$$

$$= \int_1^2 \left(x^4 - 6x + \frac{9}{x^2}\right) dx$$

$$= \left[\frac{1}{5}x^5 - 3x^2 - \frac{9}{x}\right]_1^2$$

$$= \frac{1}{5}(2^5 - 1^5) - 3(2^2 - 1^2) - 9 \cdot \left(\frac{1}{2} - 1\right)$$

$$= \frac{31}{5} - 9 + \frac{9}{2} = \frac{62 - 45}{10} = \frac{17}{10}$$

$$\Rightarrow \frac{1}{\sqrt{\frac{17}{10}}} \cdot \left(x^2 - \frac{3}{x}\right) \text{ má dĺžku } 1$$

$$\boxed{\text{ON báza: } \left(\frac{\sqrt{2}}{x}, \sqrt{\frac{10}{17}}\left(x^2 - \frac{3}{x}\right)\right)}$$

$$\begin{array}{cc} \| & \| \\ g_1 & g_2 \end{array}$$

(ii) Na V imamo

OG bazi $\left(f = \frac{1}{x},\ h = x^2 - \frac{3}{x}\right)$

ON -''- (g_1, g_2)

Najdi $p(x)$ je kolmo projekcijo funkcije x na V.

$$p(x) = \underline{\langle x, g_1 \rangle}\, g_1 + \underline{\langle x, g_2 \rangle}\, g_2$$

$$p(x) = \frac{\langle x, f \rangle}{\langle f, f \rangle} f + \frac{\langle x, h \rangle}{\langle h, h \rangle} \cdot h$$

$$\langle x, g_1 \rangle = \int_1^2 (x g_1)\, dx = \int_1^2 x \frac{\sqrt{2}}{x}\, dx$$

$$= \sqrt{2} \underbrace{\int_1^2 dx}_{=1} = \sqrt{2}$$

$$\langle x, g_2 \rangle = \int_1^2 (x g_1)\, dx = \int_1^2 \sqrt{\frac{10}{17}}\, x \left(x^2 - \frac{3}{x}\right) dx$$

$$= \sqrt{\frac{10}{17}} \int_1^2 (x^3 - 3)\, dx =$$

$$= \sqrt{\frac{10}{17}} \left[\frac{1}{4} x^4 - 3x \right]_1^2 =$$

$$= \sqrt{\frac{10}{17}} \left[\frac{1}{4} (2^4 - 1) - 3(2 - 1) \right]$$

$$= \sqrt{\frac{10}{17}} \left(\frac{15}{4} - \frac{12}{4} \right) = \frac{3}{4} \sqrt{\frac{10}{17}}$$

<u>Výsledek</u>

$$p = \sqrt{2}\, g_1 + \frac{3}{4} \sqrt{\frac{10}{17}}\, g_2 =$$

$$= \sqrt{2} \cdot \frac{\sqrt{2}}{x} + \frac{3}{4} \sqrt{\frac{10}{17}} \cdot \sqrt{\frac{10}{17}} \left(x^2 - \frac{3}{x} \right)$$

$$= 2x + \frac{15}{34} \left(x^2 - \frac{3}{x} \right)$$

(iii) Vzdálenost funkce x od podprostoru V.

$$x = \underbrace{p}_{\in V} + \underbrace{(x - p)}_{\in V^\perp}$$

Vzdálenost $= \sqrt{\langle x-p, x-p \rangle}$

$$\langle x-p, x-p \rangle = \int_1^2 \left[x - \left(2x + \frac{15}{34}\left(x^2 - \frac{3}{x}\right)\right)\right]^2 dx$$

$$= \int_1^2 \left(-x - \frac{15}{34}\left(x^2 - \frac{3}{x}\right)\right)^2 dx$$

$$= \int_1^2 \left(\frac{15}{34}\left(x^2 - \frac{3}{x}\right) - x\right)^2 dx$$

odmocnina z tohoto výrazu je hledaná vzdálenost