Matematika II

Derivace

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Operace s derivacemi

Nechť existují derivace $f'(x_0)$, $g'(x_0)$, $\varphi'(x_0)$, $f'\big(\varphi(x_0)\big)$

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$$(cf)'(x_0) = \lim_{h\to 0} \frac{cf(x_0+h) - cf(x_0)}{h} = c \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = cf'(x_0)$$

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$$
\begin{aligned}
(f+g)'(x_0) = \lim_{x\to x_0} \frac{\big(f(x)+g(x)\big)-\big(f(x_0)+g(x_0)\big)}{x-x_0} &= \\
= \lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0}\right) &= \\
= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} &= f'(x_0)+g'(x_0)
\end{aligned}
$$

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\begin{aligned}
(f-g)'(x_0) = \lim_{x\to x_0} \frac{\big(f(x)-g(x)\big) - \big(f(x_0)-g(x_0)\big)}{x-x_0} &= \\
= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0} - \frac{g(x)-g(x_0)}{x-x_0}\right) &= \\
= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} - \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} &= f'(x_0)-g'(x_0)
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$$
\begin{aligned}
(fg)'(x_0) &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = \\
&= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} = \\
&= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right) = \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0} g(x) + f(x_0) \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0)
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$$

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$$\left(\frac{1}{g}\right)'(x_0) = \lim_{x\to x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \lim_{x\to x_0} \frac{g(x_0) - g(x)}{(x - x_0)g(x)g(x_0)} =$$
$$= \lim_{x\to x_0} \left(-\frac{g(x) - g(x_0)}{x - x_0}\,\frac{1}{g(x)g(x_0)}\right) = -\frac{g'(x_0)}{g(x_0)^2}$$

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$$\left(\frac{f}{g}\right)'(x_0) = \left(f\frac{1}{g}\right)'(x_0) = f'(x_0)\left(\frac{1}{g}\right)(x_0) + f(x_0)\left(\frac{1}{g}\right)'(x_0) =$$

$$= \frac{f'(x_0)}{g(x_0)} - f(x_0)\frac{g'(x_0)}{g(x_0)^2} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

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$$
\begin{aligned}
(f \circ \varphi)'(x_0) &= \lim_{x\to x_0} \frac{f\big(\varphi(x)\big) - f\big(\varphi(x_0)\big)}{x - x_0} = \\
&= \lim_{x\to x_0} \left(\frac{f\big(\varphi(x)\big) - f\big(\varphi(x_0)\big)}{\varphi(x) - \varphi(x_0)} \, \frac{\varphi(x) - \varphi(x_0)}{x - x_0} \right) = \\
&= \lim_{x\to x_0} \frac{f\big(\varphi(x)\big) - f\big(\varphi(x_0)\big)}{\varphi(x) - \varphi(x_0)} \lim_{x\to x_0} \frac{\varphi(x) - \varphi(x_0)}{x - x_0} = f'\big(\varphi(x_0)\big)\varphi'(x_0)
\end{aligned}
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- $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$
- $(f \circ \varphi)'(x_0) = f'\big(\varphi(x_0)\big)\varphi'(x_0)$

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- $(f \circ \varphi)'(x_0) = f'\big(\varphi(x_0)\big)\varphi'(x_0)$

$$
\begin{aligned}
f^{-1}(x_0) &= y_0 \\
x_0 &= f(y_0) \\
x_0 &= f\big(f^{-1}(x_0)\big) \\
1 &= f'\big(f^{-1}(x_0)\big)\big(f^{-1}\big)'(x_0) \\
\frac{1}{f'\big(f^{-1}(x_0)\big)} &= \big(f^{-1}\big)'(x_0)
\end{aligned}
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- $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$
- $(f \circ \varphi)'(x_0) = f'\big(\varphi(x_0)\big)\varphi'(x_0)$
- $\left(f^{-1}\right)\prime(x_0) = \dfrac{1}{f'\big(f^{-1}(x_0)\big)} = \dfrac{1}{f'(y_0)}$

Derivace elementárních funkcí

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\operatorname{tg} x$	$\frac{1}{(\cos x)^2}$
x^n	nx^{n-1}	$\operatorname{cotg} x$	$-\frac{1}{(\sin x)^2}$
e^x	e^x	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\ln x$	$\frac{1}{x}$	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\operatorname{arccotg} x$	$-\frac{1}{1+x^2}$
$\sin x$	$\cos x$	$\ln\left(x \pm \sqrt{1+x^2}\right)$	$\pm\frac{1}{\sqrt{1+x^2}}$
$\cos x$	$-\sin x$	$\ln\sqrt{\frac{1+x}{1-x}}$	$\frac{1}{1-x^2}$

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